ON THE HIGH FREQUENCY ASYMPTOTIC EVALUATION OF THE POTENTIALS OF ELEMENTAL SOURCES ON AN ANISOTROPIC IMPEDANCE CYLINDER

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Chart 1: The motivation for the work to be described arose because of a desire to more fully understand the coupling between antennas mounted on airframes. A computational model existed which could assess the coupling on a conducting surface by means of the Geometrical Theory of Diffraction. However, modern airframes include non-perfectly conducting and non-conducting materials. The present work makes use of the formulation of Pearson [Radio Science, 1986] [IEEE Trans. AP-35, June 1987] for the multilayered cylinder problem to obtain an asymptotic representation of the coupling in the presence of these more general materials.

Chart 2: Pearson treated the problem of computing the field of an elemental source radiating in the presence of a multilayered circular cylinder. As shown, the source and field points need not be a t the same axial coordinate so that the ray path is not circular but helical.

Chart 3: Asymptotic evaluation of Pearson's expression for the potentials generated by the source in the presence of the multilayered cylinder are far too complicated to evaluate asymptotically. However, at any given fixed complex azimuthal mode order, ν, one may compute an anisotropic surface impedance which models the reflection properties of the cylinder exactly. At nearby values of the order, the result is approximately correct. The expression for the potentials in the presence of the impedance cylinder is simple enough to treat asymptotically and that is what is described here.

Chart 4: (Vector Potentials - I) This is Pearson's expression for the surface vector potentials generated by axially directed electric and magnetic elemental sources located on the surface of a cylinder. The discrete sum denotes accumulation of the terms corresponding to multiple circumnavigations of the cylinder. Note that this is a double spectral integral over the complex azimuthal mode order, v, and-the axial wavenumber, α. Defining the logarithmic derivative, L, and the ratio of Hankel functions, Q, one may express the elements of the 2X2 reflection matrix, R_B, in the form shown in the next chart.

Chart 5: (Vector Potentials - II) The elements of the 2X2 reflection matrix in the integrand of the spectral integral represent the coupling between the axially directed sources and the vector potentials due to the reflecting cylindrical impedance surface. The elements of the matrix may be expressed in terms of the elements of the 2X2 impedance matrix of the reflecting surface in the form shown in this chart. Note that the denominator, A, is of the same order of complexity as the numerators.

Chart 6: (Vector Potentials - III) Substituting the explicit expressions for the reflection matrix elements into the integrand and simplifying by means of the Wronskian of the Hankel functions results in these expressions for the vector potentials. We now proceed to carry out the integration over the axial wavenumber by the method of stationary phase.

Chart 7: (Stationary Phase Result - I) The result of the stationary phase integration may be expressed in terms of Fock type integrals denoted here by v. D is the arc length of the ray path on the cylinder and θ is the angle between the path and the axial direction. (For and azimuthal path, θ is 90 degrees.)

Chart 8: (Stationary Phase Result - II) The Fock type integrals, v, may be expressed in the form shown here. Note that the v integration variable has been replaced by τ via the change of variable shown at the bottom of the chart. Similarly the azimuthal separation of the source and field points is now represented by the so-called Fock parameter, ξ, defined as indicated. Our goal is to evaluate these integrals asymptotically for electrically large radius cylinders.

Chart 9: (Asymptotic Evaluation - I) Following the work of many earlier researchers [e.g. , Paknys and Wang, IEEE Trans. AP-35, March 1987] we approximate the logarithmic derivative, L, using the fock type Airy functions, $w(\tau)$. Of course, the large ξ behavior of the integrals is accessible via the familiar residue series of creeping waves. However, the small, ξ behavior, corresponding to large τ, is more difficult to obtain. Again following earlier work, we expand the corresponding ratio, w'/w, for large τ as shown.

Chart 10: (Asymptotic Evaluation - II) In terms of this ratio, $\mathfrak{R}$, the denominator of the integrands takes the form indicated here. Now, for the azimuthal case treated by earlier workers, the last product disappears and one of the factors of the first product is canceled by a similar factor in the numerator resulting in an integrand with a single binomial in the denominator. This type of integrand has been treated effectively by Wait [J. Res. NBS, April 1956], by Bremmer [IRE Trans., AP-9, July 1958], and by Hill and Wait [Radio Science, May-June 1980]. Thus , if we can express our more general integrand as a sum of these previously treated integrands, the problem is solved. We proceed as follows.

First, reverse the expansion of $\mathfrak{R}$ for large τ to yield a series for τ as a function of $\mathfrak{R}$ as shown. Then note that the previous denominator expression multiplied by $\mathfrak{R}^2$ is a polynomial in $\mathfrak{R}$. In this case we have retained only the first two terms of the τ expansion so the polynomial is of sixth degree. Three terms would have resulted in a polynomial of twelfth degree. Finally, factor the polynomial. into binomial factors as indicated at the bottom the chart.

Chart 11: (Asymptotic Evaluation - III) Our Fock type integrals now take the form shown which is easily recognized as a sum of six integrals of the type previously treated in the literature. Following these treatments, we substitute the large τ expansion of $\mathfrak{R}$ displayed at the bottom of the chart thereby reducing the integral to a sum of known Laplace transforms. If q is small in magnitude, one expands the integrand (exclusive of the exponential) in a series of inverse powers of τ. The transform then results in a series of positive powers of ξ representing v. If the magnitude of q is large, a more effective approach due to Wait and Bremmer is to expand in terms of the Laplace transform of the complementary error function, $\mathrm{erfc}(q\sqrt{\tau})$.

Chart 12: (Asymptotic Evaluation Summary) In summary, then, for large separation of source and field points (large Fock parameter, ξ), one uses the residue series. for small Fock parameter, one uses a power series for small q. For large q, however, one uses the complementary error function representation. Note that the desired Fock integral is only equal to the expression shown for certain ranges of q. In other ranges one must use a different expression obtained by changing the sign of q.

Chart 13: A few selected examples will be discussed as time permits.

Last Chart: (Concluding Summary) The azimuthal case where θ is 90 degrees has been treated successfully in the literature. It was determined that for large Fock parameter the creeping wave (residue) series is useful. For small Fock parameter, however, this series is very slowly convergent and, therefore, impractical. In this case, one expands the integrand (exclusive of the exponential) in terms of known Laplace transforms. These approaches have all been generalized to the nonazimuthal case here although it must be noted that, because of the nature of the stationary phase integration applied to the axial. wave number, θ equal to zero (axial paths) must be excluded. The generalization was accomplished by reducing the general. case to a summation of integrals of the form arising in the simpler previously treated azimuthal case.

On the High Frequency Asymptotic Evaluation of the Potentials of Elemental Sources on an Anisotropic Impedance Cylinder

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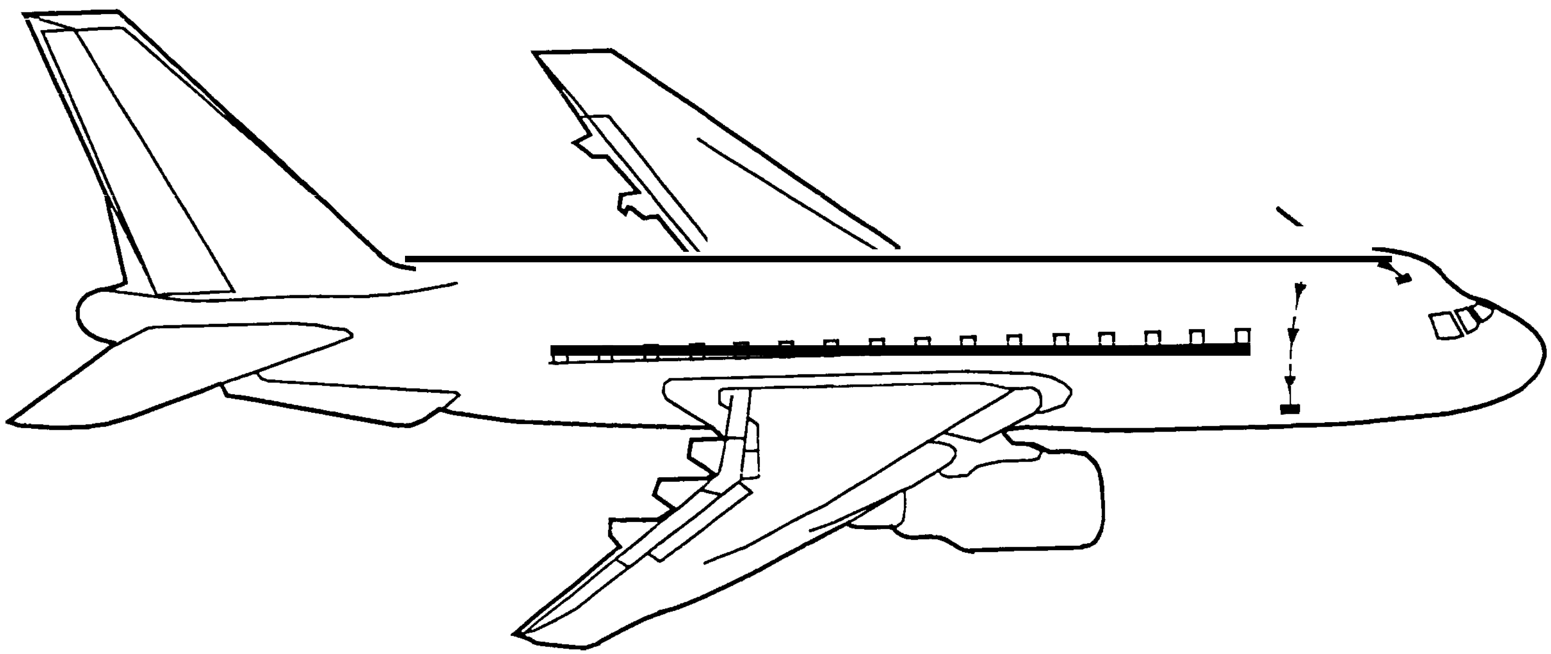
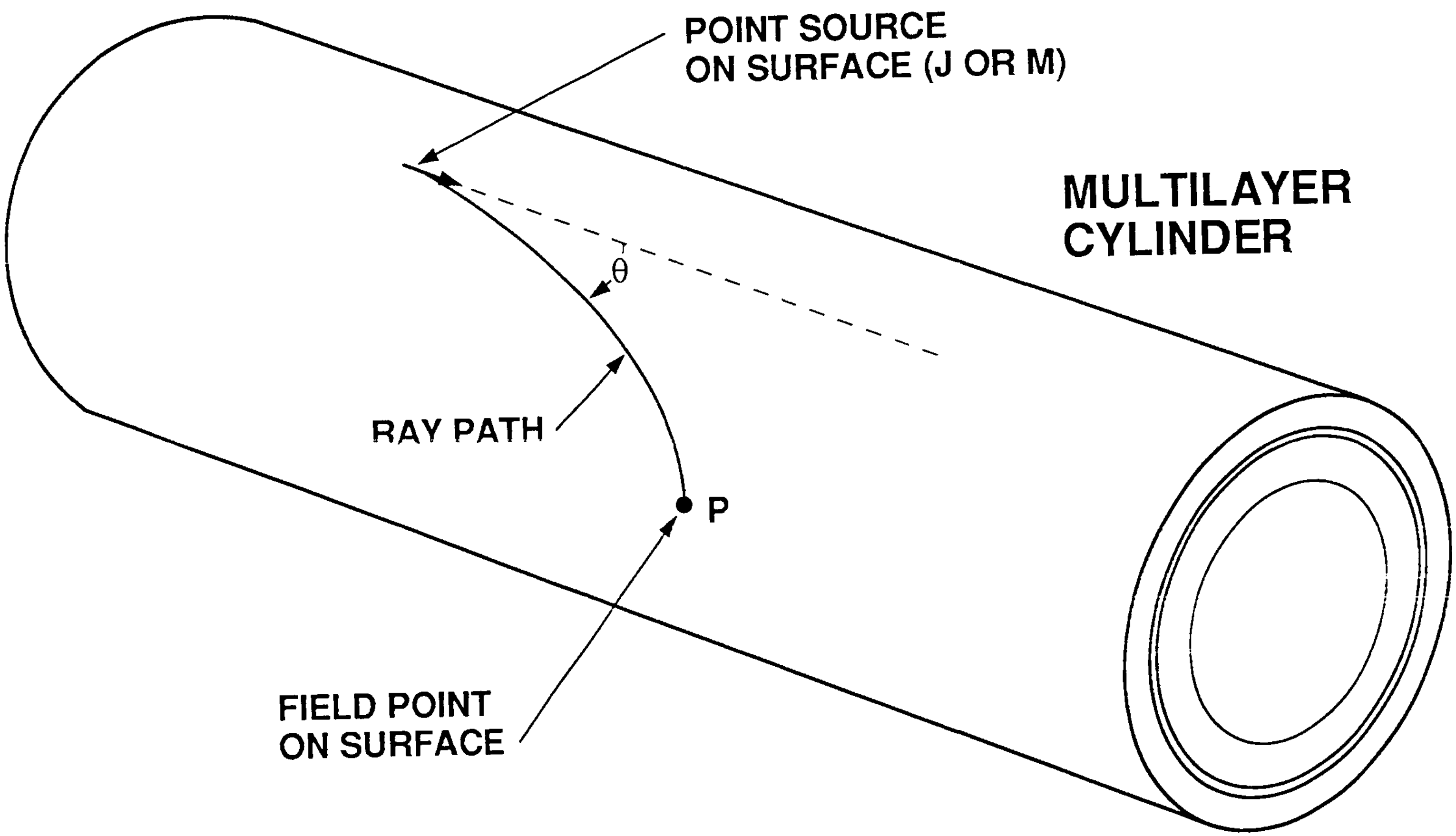
ANTENNA COUPLING ON AN AIRFRAME

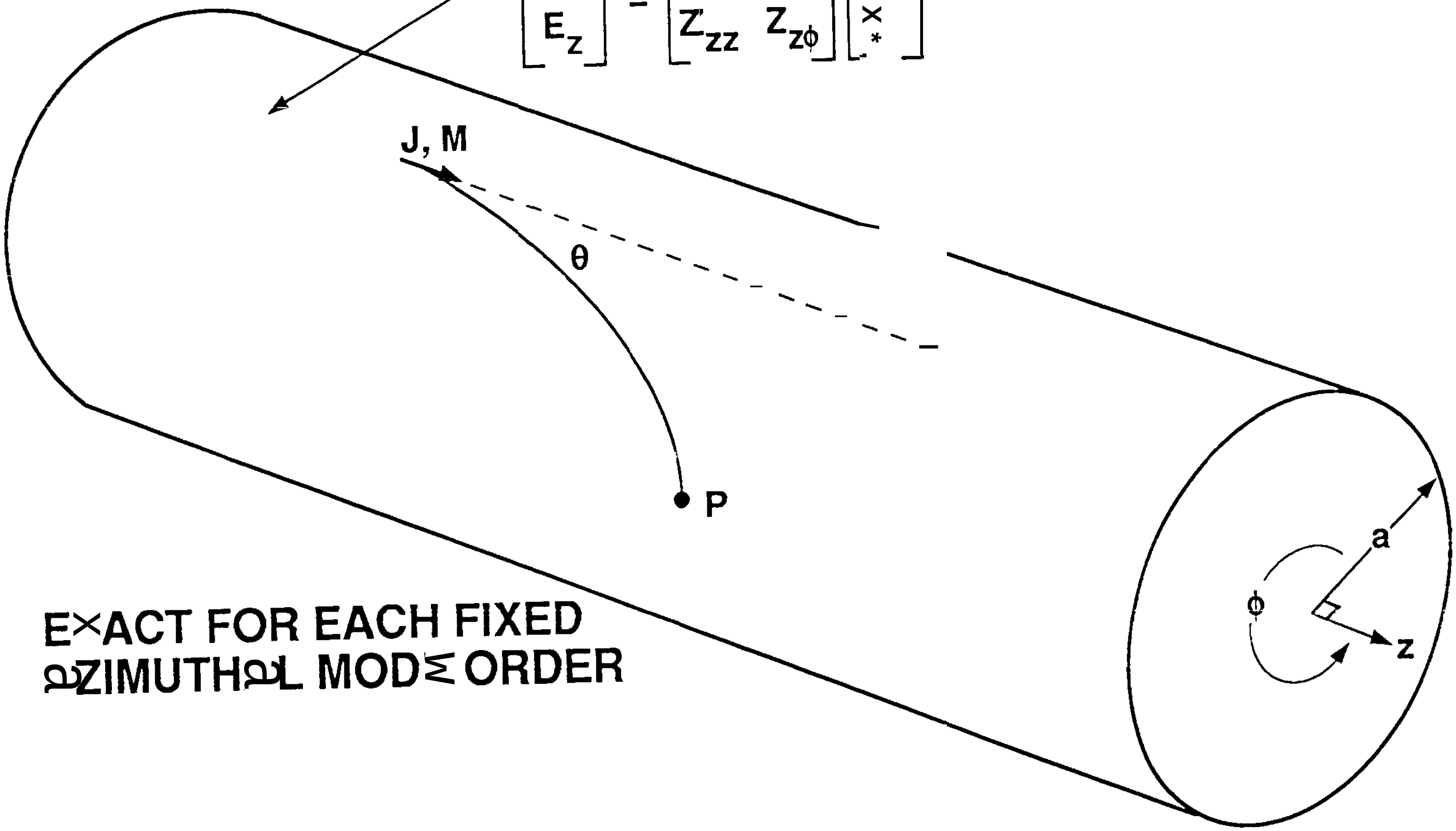


COUPLING GEOMETRY



SURFACE IMPEDANCE MODEL

$$\begin{bmatrix} E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} Z_{\phi z} & Z_{\phi\phi} \\ Z_{zz} & Z_{z\phi} \end{bmatrix} \begin{bmatrix} H_z \\ \times_* \end{bmatrix}$$



EXACT FOR EACH FIXED
AZIMUTHAL MODE ORDER

Vector Potentials - I

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{F} \end{bmatrix} = \frac{j}{16\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{\nu}^{(2)}(\beta_1 a) \left\{ H_{\nu}^{(1}\ \beta_1 a)\ {=}I{=} + H_{\nu}^{(2)}(\beta_1 a)\ {=}R_B(\nu,\alpha)\,] \right\}$$

$$\times \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix} e^{-j\alpha(z-z')}\ e^{-j\nu|\phi-\phi_n'}\ d\nu\ d\alpha$$

where $\phi_n' = \phi' + 2n\pi$ and $\beta_1 = \sqrt{k_1^2 - \alpha^2}$

$$L_{ij} = \frac{jk_j\ H_{\nu}^{(i\,\prime}(\beta_j a)}{\beta_j\ H_{\nu}^{(i)}\ \beta_j a)} \qquad Q_{ij}(z) = \frac{H^{\prime\,i)}(z)}{H^{\prime\,j)}(z)}$$

Vector Potentials - II

$$R_{B_{11}}=\left\{\left[\frac{\Delta_z}{Z_{z\phi}}-\eta_1 L_{11}\right]\left[\frac{L_{21}}{\eta_1}+\frac{1}{Z_{z\phi}}\right]-\left[\frac{\alpha\nu}{a\beta_1^2}-\frac{Z_{zz}}{Z_{z\phi}}\right]\left[\frac{\alpha\nu}{a\beta_1^2}+\frac{Z_{\phi\phi}}{Z_{z\phi}}\right]\right\}\frac{Q12}{\Delta}$$

$$R_{B_{12}}=\left\{\left[\frac{L_{21}}{\eta_1}-\frac{L_{11}}{\eta_1}\right]\left[\frac{\alpha\nu}{a\beta_1^2}+\frac{Z_{\phi\phi}}{Z_{z\phi}}\right]\right\}\frac{Q_{12}}{\Delta}\eta_1^2$$

$$R_{B_{21}}=\left\{\left[\eta_1 L_{11}-\eta_1 L_{21}\right]\left[\frac{\alpha\nu}{a\beta_1^2}-\frac{Z_{zz}}{Z_{z\phi}}\right]\right\}\frac{Q_{12}}{\eta_1^2\Delta}$$

$$R_{B_{22}}=\left\{\left[\frac{\Delta_z}{Z_{z\phi}}-\eta_1 L_{21}\right]\left[\frac{L_{11}}{\eta_1}+\frac{1}{Z_{z\phi}}\right]-\left[\frac{\alpha\nu}{a\beta_1^2}-\frac{Z_{zz}}{Z_{z\phi}}\right]\left[\frac{\alpha\nu}{a\beta_1^2}+\frac{Z_{\phi\phi}}{Z_{z\phi}}\right]\right\}\frac{Q_{12}}{\Delta}$$

$$\Delta=\left\{\left[\frac{\Delta_z}{Z_{z\phi}}-\eta_1 L_{21}\right]\left[-\frac{L_{21}}{\eta_1}-\frac{1}{Z_{z\phi}}\right]+\left[\frac{\alpha\nu}{a\beta_1^2}-\frac{Z_{zz}}{Z_{z\phi}}\right]\left[\frac{\alpha\nu}{a\beta_1^2}+\frac{Z_{\phi\phi}}{Z_{z\phi}}\right]\right\}$$

where $\Delta_z = Z_{\phi z}Z_{z\phi} - Z_{\phi\phi}Z_{zz}$, $\eta_1=\sqrt{\frac{\mu_1}{\varepsilon_1}}$,

Vector Potentials - III

$$\mathbf{A} = \frac{j}{16\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4k_1}{\pi\beta_1 a} \frac{1}{\Delta} \left[L_{21} + \frac{\eta_1}{Z_{z\phi}} \right] \mathbf{J}\, e^{-j\alpha(z-z')}\, e^{-j\nu|\phi-\phi_n'|}\, d\nu\, d\alpha$$

$$+ \frac{j}{16\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4k_1}{\pi\beta_1 a} \frac{1}{a} \left[\frac{\alpha\nu}{a\beta_1^2} + \frac{Z_{\phi\phi}}{Z_{z\phi}} \right] \mathbf{M}\, e^{-j\alpha(z-z')}\, e^{-j\nu|\phi-\phi_n'|}\, d\nu\, d\alpha$$

$$\mathbf{F} = \frac{j}{16\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4k_1}{\pi\beta_1 a} \frac{1}{\Delta} \left[L_{21} - \frac{Z_{\phi\phi}}{\eta_1 Z_{z\phi}} \right] \mathbf{M}\, e^{-j\alpha(z-z')}\, e^{-j\nu|\phi-\phi_n'|}\, d\nu\, d\alpha$$

$$+ \frac{j}{16\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4k_1}{\pi\beta_1 a} \frac{1}{\Delta} \left[\frac{\alpha\nu}{a\beta_1^2} - \frac{Z_{zz}}{Z_{z\phi}} \right] \mathbf{J}\, e^{-j\alpha(z-z')}\, e^{-j\nu|\phi-\phi_n'|}\, d\nu\, d\alpha$$

Stationary Phase Result - I

$$\mathbf{A} = \frac{1}{8}\, e^{j\pi/4}\, \frac{e^{-jk_1 D}}{\sqrt{k_1 D}} \sqrt{\frac{2}{\xi}}\, k_1 \cos\theta \,[\mathbf{J}\, v_{11} + \mathbf{M}\, v_{12}]$$

$$\mathbf{F} = \frac{1}{8}\, e^{j\pi/4}\, \frac{e^{-jk_1 D}}{\sqrt{k_1 D}} \sqrt{\frac{2}{\xi}}\, k_1 \cos\theta \,[\mathbf{M}\, v_{22} + \mathbf{J}\, v_{21}]$$

where

$$D = \sqrt{(z-z')^2 + a^2(\phi-\phi')^2} \text{ and } \theta = \arctan\left[\frac{z-z'}{a(\phi-\phi')}\right]$$

Stationary Phase Result - II

$$v_{11}(\xi) = \frac{4k_1 m}{\pi\beta_1^2 a}\,\frac{1}{2}\,e^{j\pi/4}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\left[L_{21} + \frac{\eta_1}{Z_{z\phi}}\right]\,\text{t-ha}\;e^{-j\xi\tau}\,d\tau$$

$$v_{12}(\xi) = \frac{4k_1 m}{\pi\beta_1^2 a}\,\frac{1}{2}\,e^{j\pi/4}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\left[\frac{\alpha\nu}{a\beta_1^2} + \frac{Z_{\phi\phi}}{Z_{z\phi}}\right]\frac{1}{\Delta}\,e^{-j\xi\tau}\,d\tau$$

$$v_{21}(\xi) = \frac{4k_1 m}{\pi\beta_1^2 a}\,\frac{1}{2}\,e^{j\pi/4}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\left[\frac{\alpha\nu}{a\beta_1^2} - \frac{Z_{zz}}{Z_{z\phi}}\right]\frac{1}{\Delta}\,e^{-j\xi\tau}\,d\tau$$

$$v_{22}(\xi) = \frac{4k_1 m}{\pi\beta_1^2 a}\,\frac{1}{2}\,e^{j\pi/4}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\left[L_{21} - \frac{\Delta_z}{\eta_1 Z_{z\phi}}\right]\frac{1}{\Delta}\,e^{-j\xi\tau}\,d\tau$$

$m = (\beta_1 a/2)^{1/3}$, $\nu = m\tau + \beta a$, $\xi = m|\phi - \phi'|$

Asymptotic Evaluation - I

$$L_{ij} = \frac{jk_j \; H_\nu^{(i)\prime}(\beta_j a)}{\beta_j \; H_\nu^{(i)}(\beta_j a)} \approx -\frac{jk_j}{m\beta_j} \frac{w_2'(\tau)}{w_2(\tau)}$$

where $w_2(\tau) = \sqrt{\pi}\,[B_i(\tau) - j\,A_i(\tau)]$

and A_i and B_i are Airy functions

$$\Re = \frac{w_2'}{w_2} \sim \sqrt{\tau} - \frac{1}{4\tau} - \frac{5}{32\tau^{5/2}} - \frac{15}{64\tau^4} - \cdots$$

Asymptotic Evaluation - II

$$\Delta \sim (\Re + C_1)(\Re + C_2) + (\tau + C_3)\ \tau + C_4)$$

$$\tau \sim \Re^2 + \frac{1}{2\Re} + \frac{1}{8\Re^4} + \ldots$$

$$\Re^2\Delta \sim C_o\ (\Re - q_1)(\Re - q_2)(\Re - q_3)(\Re - q_4)(\Re - q_5)(\Re - q_6)$$

Asymptotic Evaluation - III

$$\frac{f(\Re)}{\Delta} = \sum_{n=1}^{6} \frac{A_n}{(\Re - q_n)}$$

$$v \sim \sum_{n=1}^{6} \int_{-\infty}^{\infty} \frac{A_n}{(\Re - q_n)} e^{-j\xi\tau} \, d\tau$$

$$\Re = \frac{w_2'}{w_2} \sim \sqrt{\tau} - \frac{1}{4\tau} - \frac{5}{32\tau^{5/2}} - \frac{15}{64\tau^4} - \cdots$$

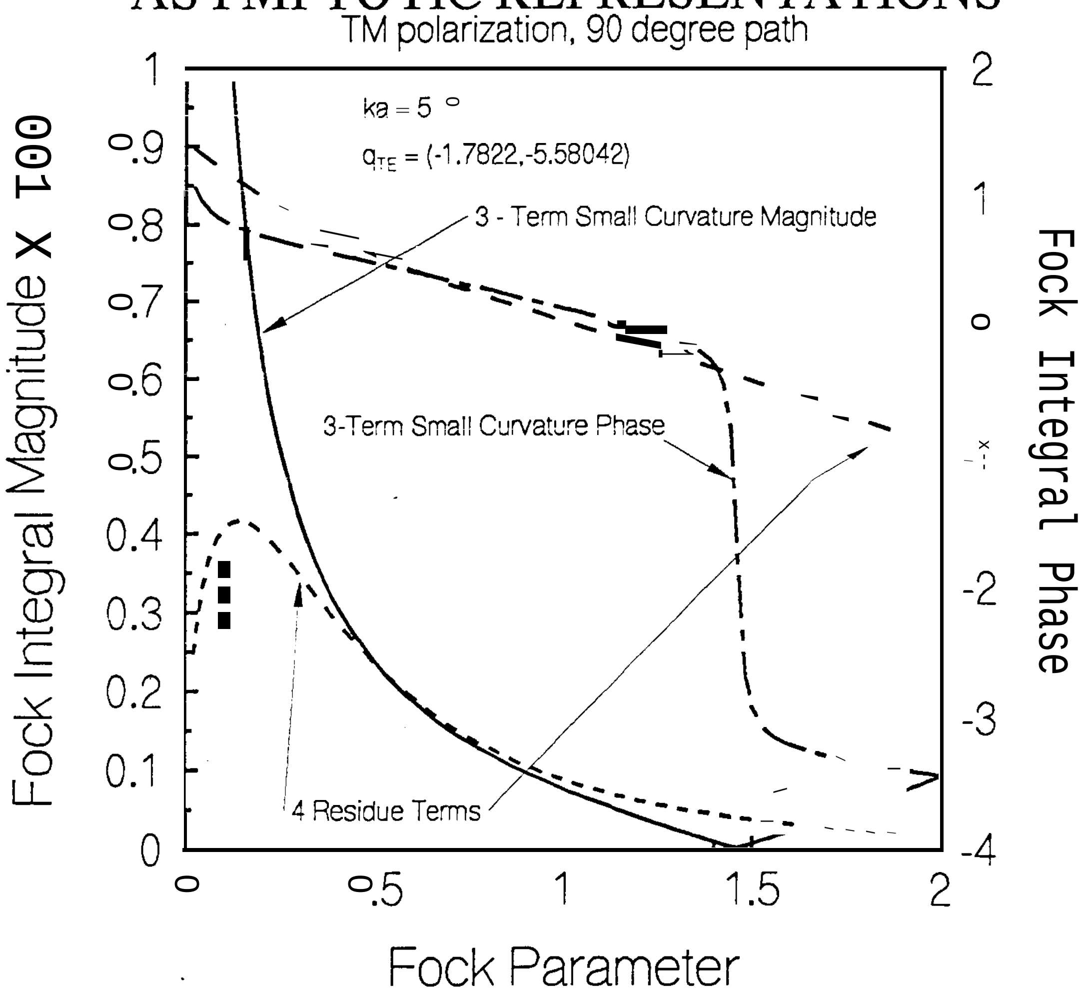
Asymptotic Evaluation Summary

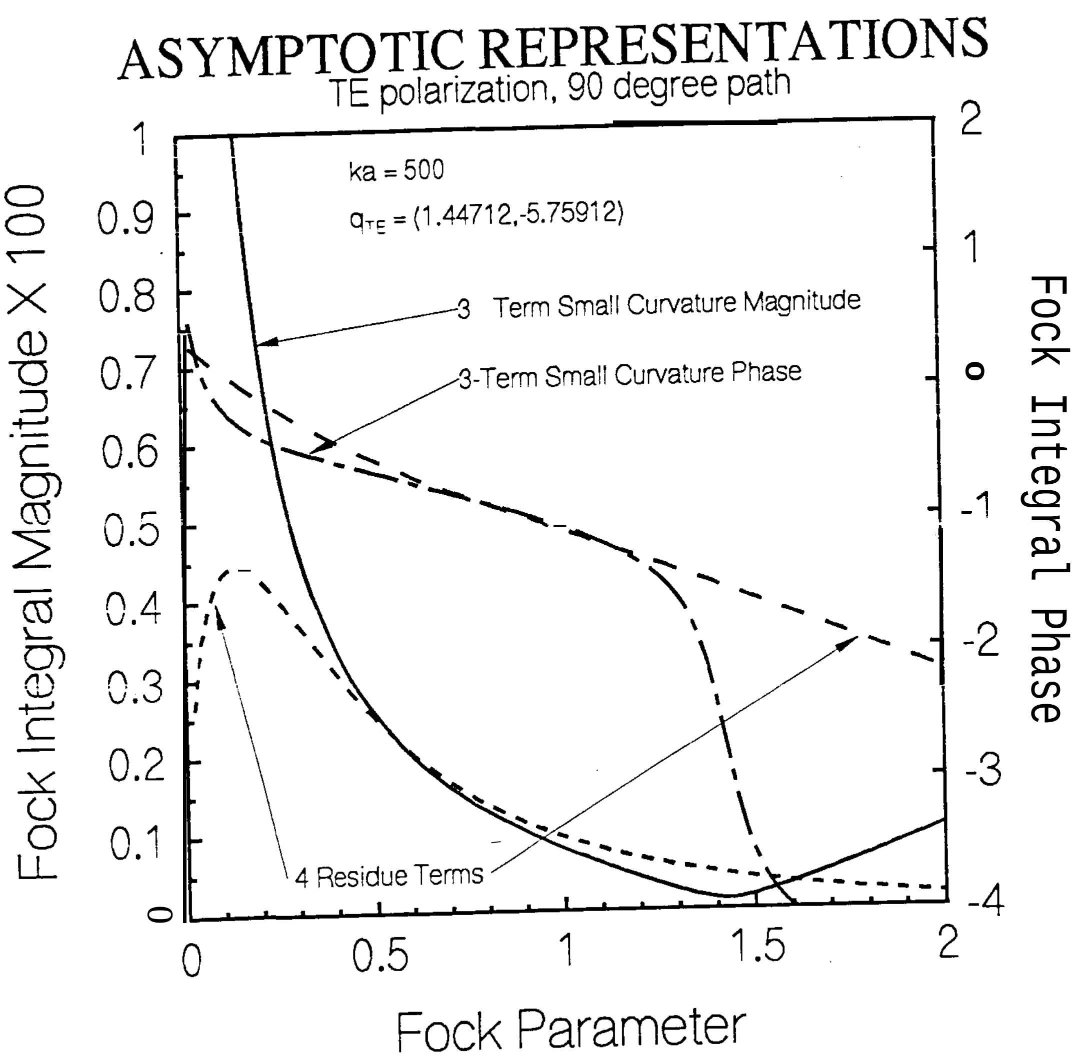
- **Large ξ: Residue series**
- **Small ξ: Power series in inverse powers of τ leading to a series of positive powers of ξ**
- **If q is large, the power series is poorly convergent unless ξ is extremely small.**
- **For large q, Wait and Bremmer obtain a "small curvature" approximation via the Laplace transform pair:**

$$\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{1}{(\sqrt{s}+q)}e^{st}\,ds=\frac{1}{\sqrt{\pi t}}-q\,e^{q^2t}\,\mathrm{erfc}\,q\sqrt{t}$$

EXAMPLES

ASYMPTOTIC REPRESENTATIONS

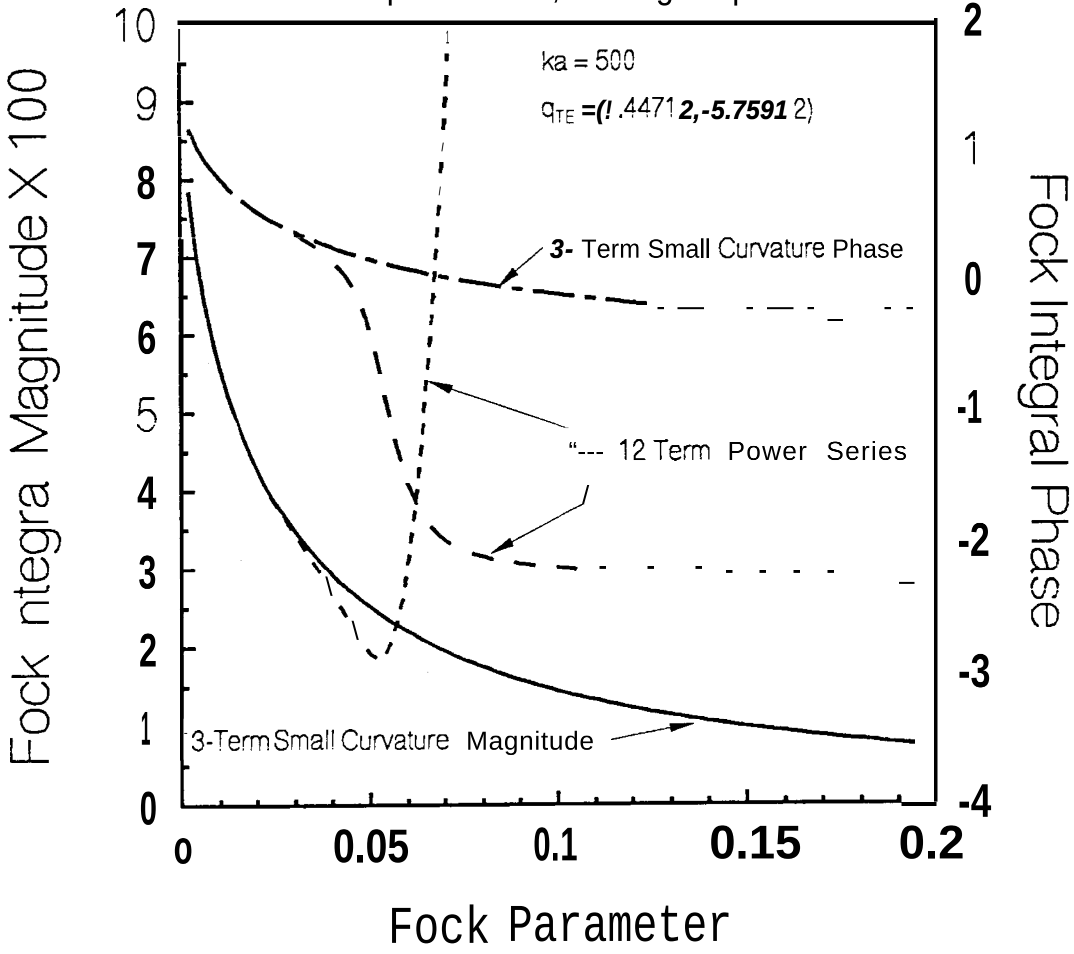




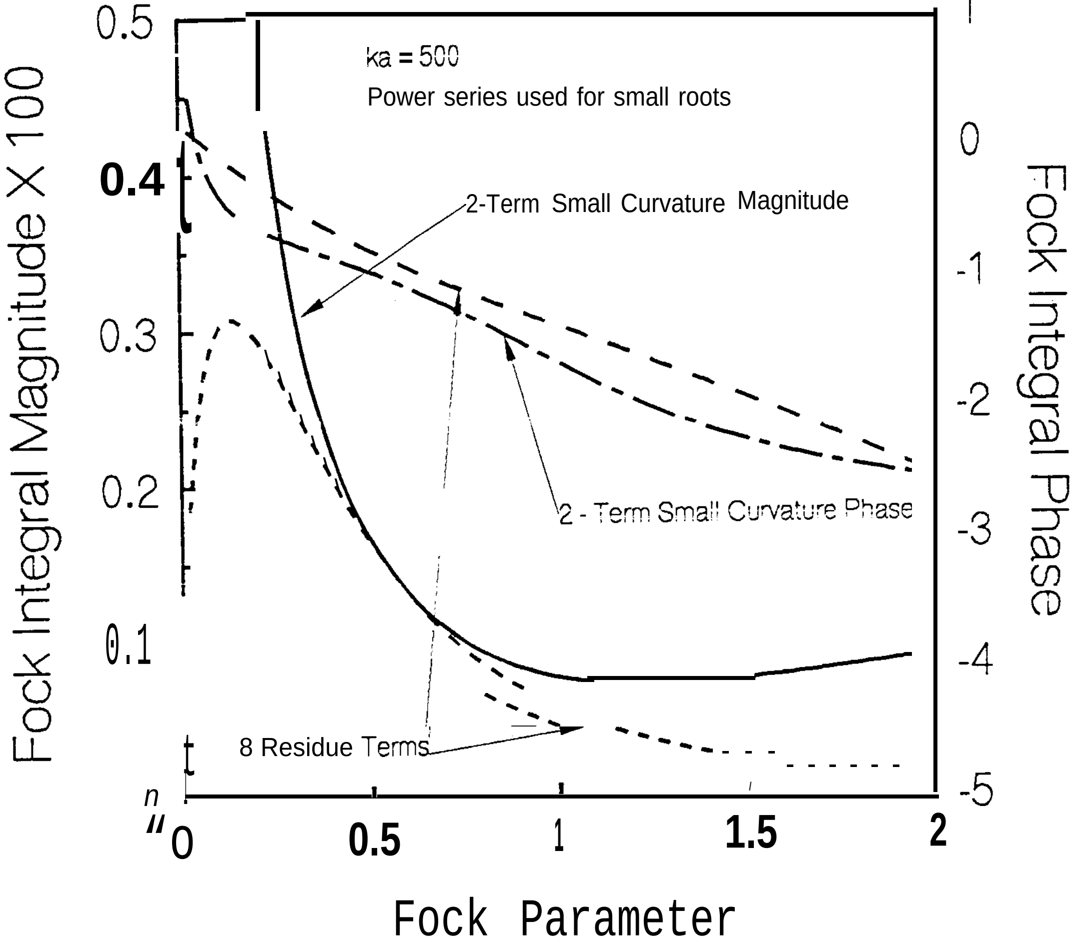
ASYMPTOTIC REPRESENTATIONS
TE polarization, 90 degree path
ka = 500
q_TE = (1.44712,-5.75912)
3 Term Small Curvature Magnitude
3-Term Small Curvature Phase
4 Residue Terms
Fock Integral Magnitude X 100
Fock Integral Phase
Fock Parameter
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
-4
-3
-2
-1
o
1
2

ASYMPTOTIC REPRESENTATIONS

TE polarization, 90 degree path

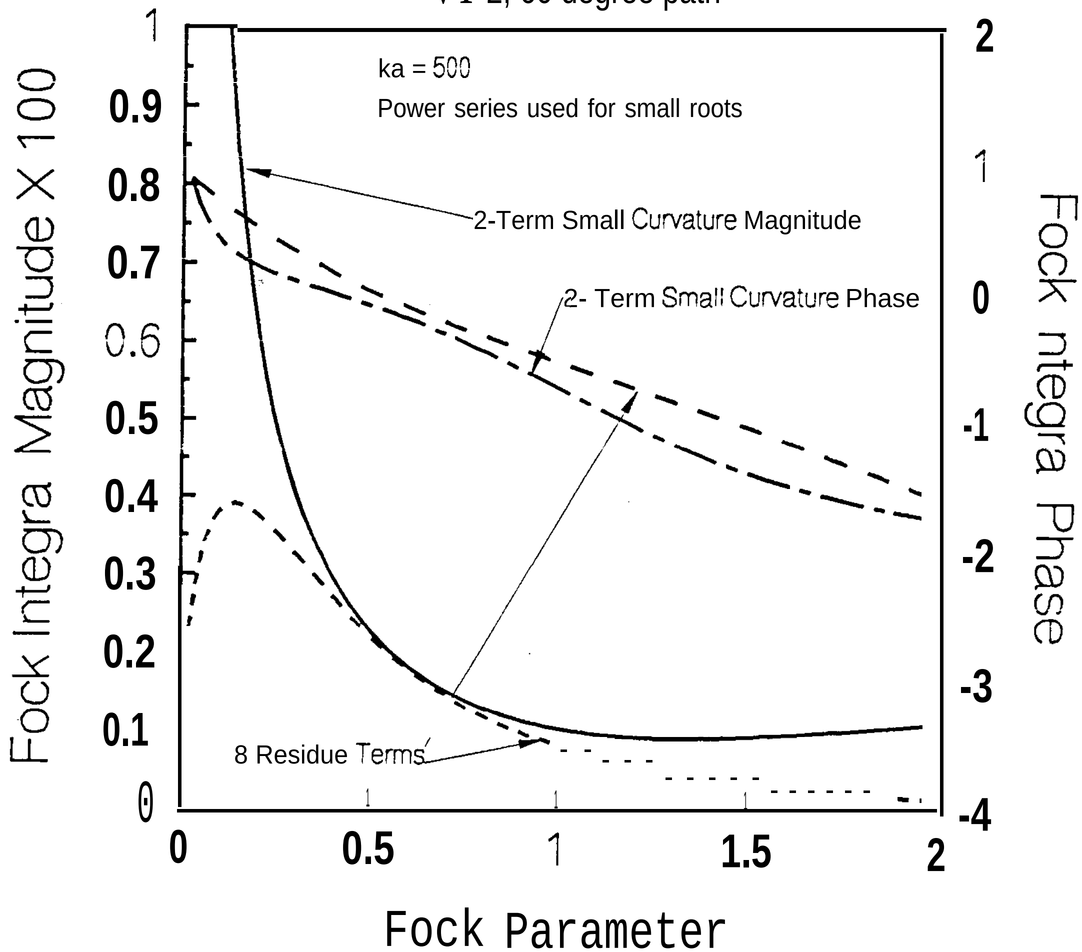


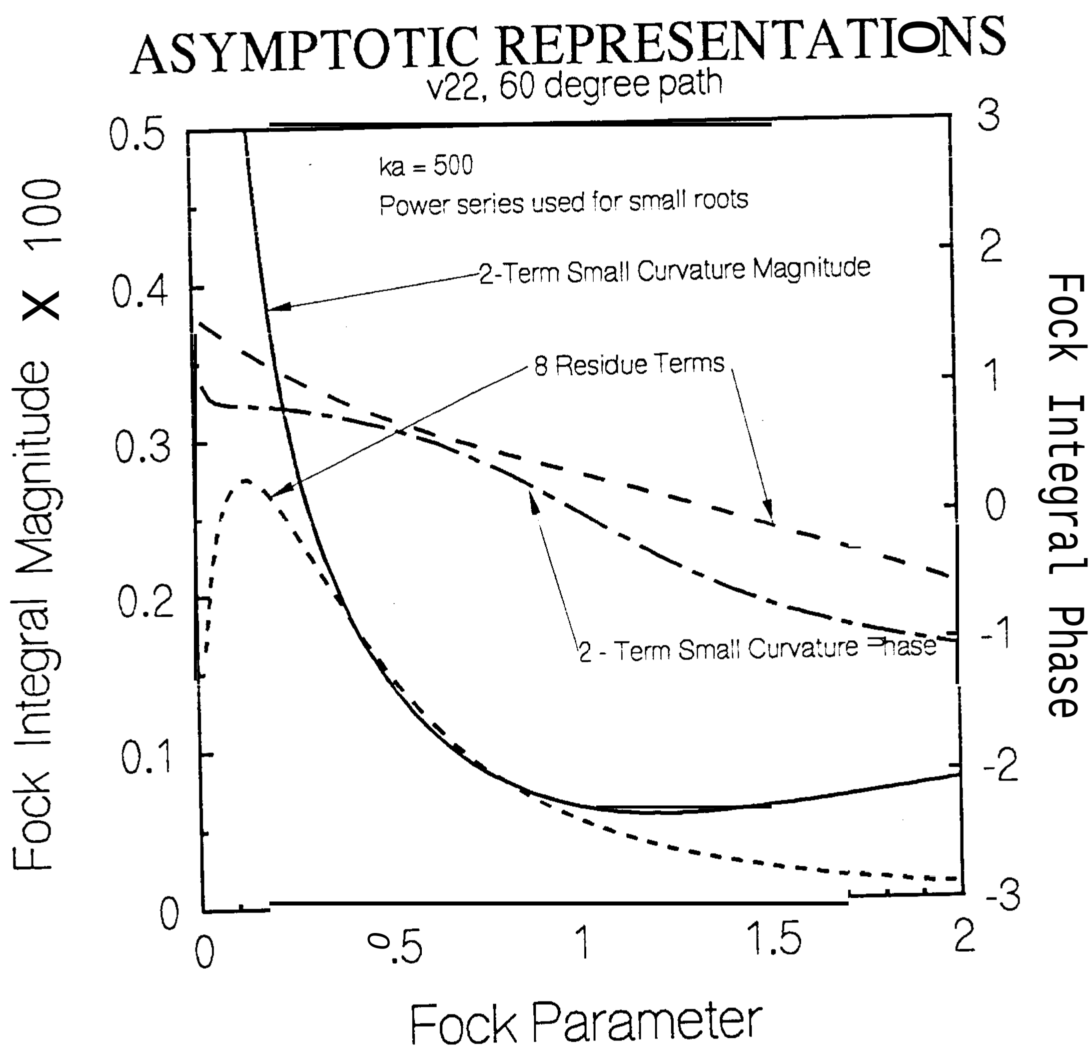
VI 1, 60 degree path

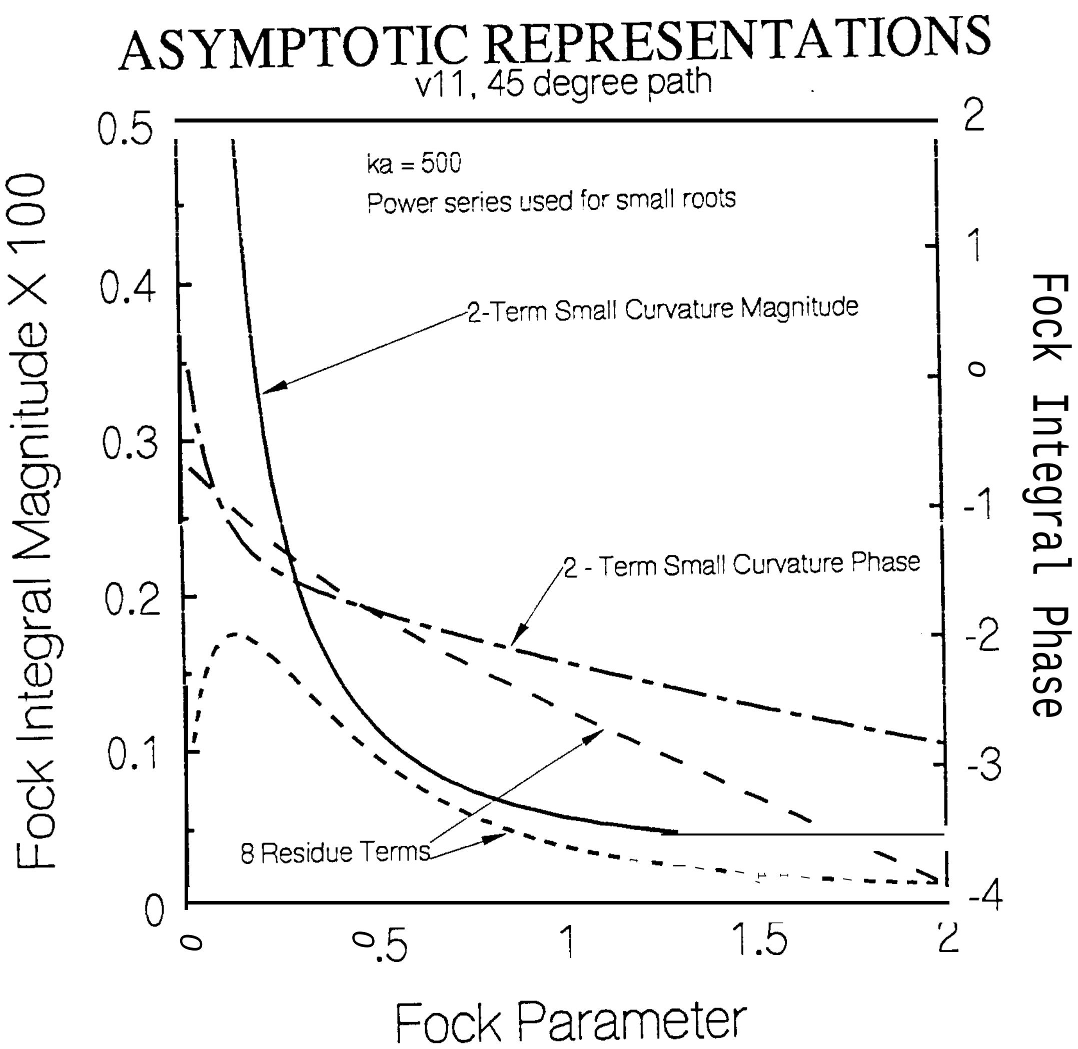


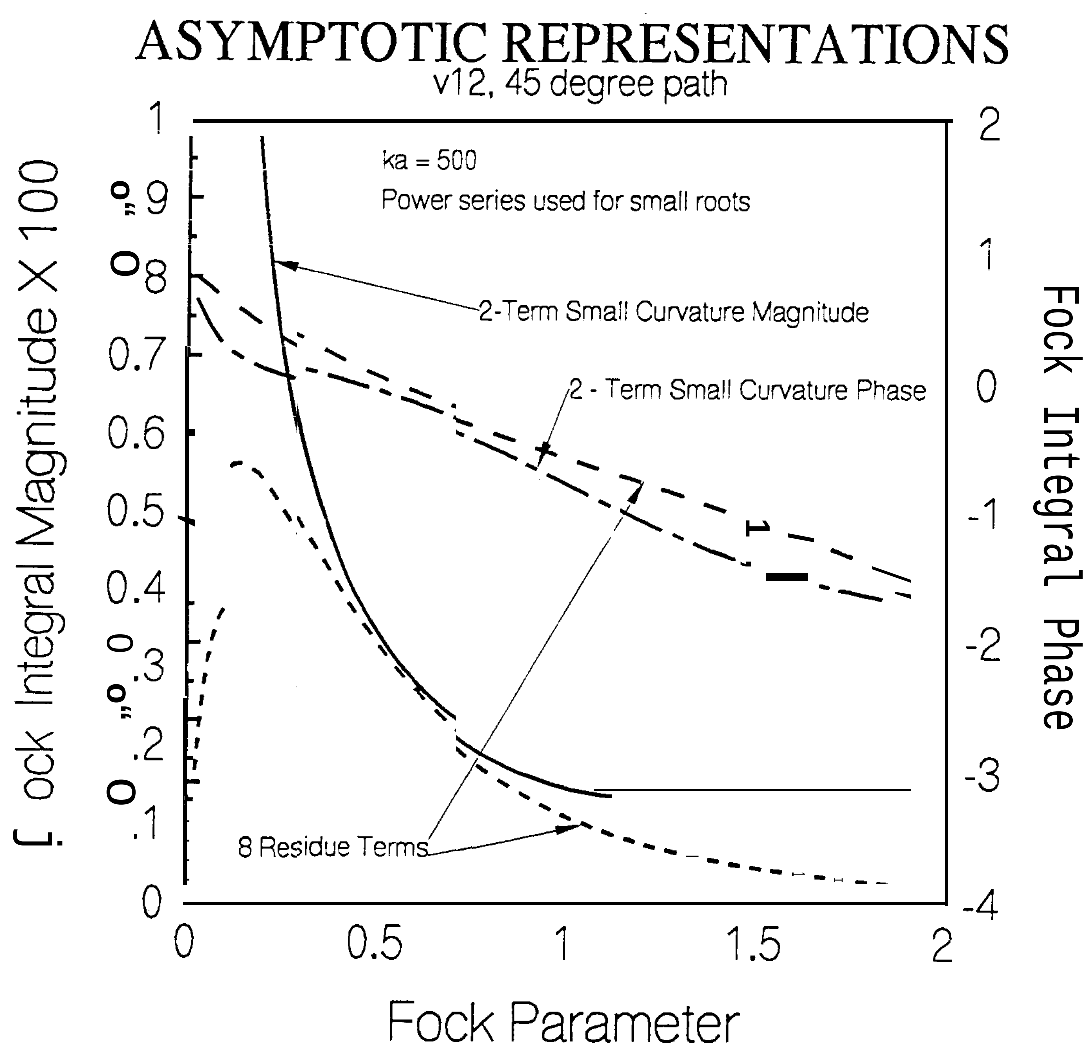
ASYMPTOTIC REPRESENTATIONS

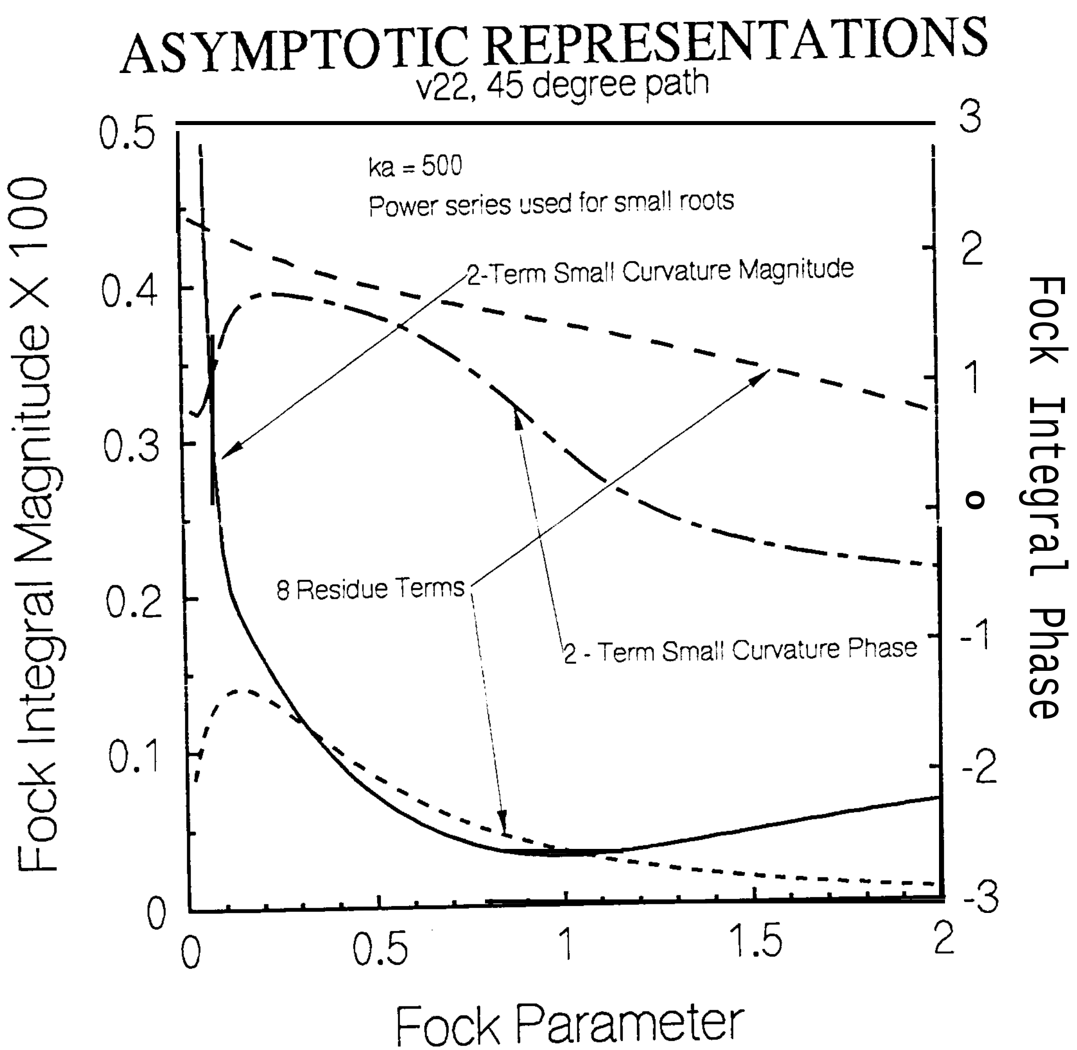
VI 2, 60 degree path




ASYMPTOTIC REPRESENTATIONS
v22, 60 degree path
ka = 500
Power series used for small roots
2-Term Small Curvature Magnitude
8 Residue Terms
2 - Term Small Curvature Phase
Fock Integral Magnitude X 100
Fock Integral Phase
Fock Parameter
0.5
0.4
0.3
0.2
0.1
0
3
2
1
0
-1
-2
-3
0
.5
1
1.5
2


ASYMPTOTIC REPRESENTATIONS
v11, 45 degree path
ka = 500
Power series used for small roots
2-Term Small Curvature Magnitude
2 - Term Small Curvature Phase
8 Residue Terms
Fock Integral Magnitude X 100
Fock Integral Phase
Fock Parameter
0.5
0.4
0.3
0.2
0.1
0
2
1
o
-1
-2
-3
-4
0
0.5
1
1.5
2

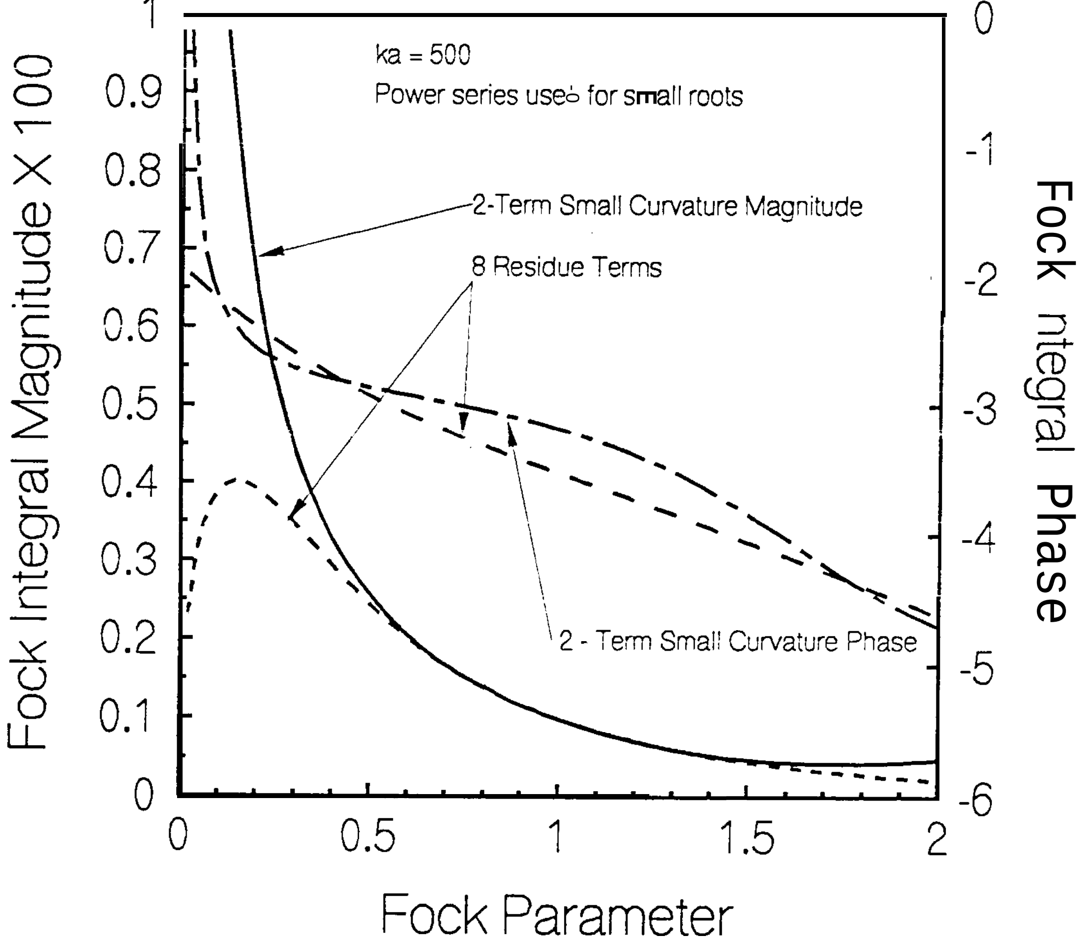

ASYMPTOTIC REPRESENTATIONS
v12, 45 degree path
ka = 500
Power series used for small roots
2-Term Small Curvature Magnitude
2 - Term Small Curvature Phase
8 Residue Terms
Fock Integral Magnitude X 100
Fock Integral Phase
Fock Parameter

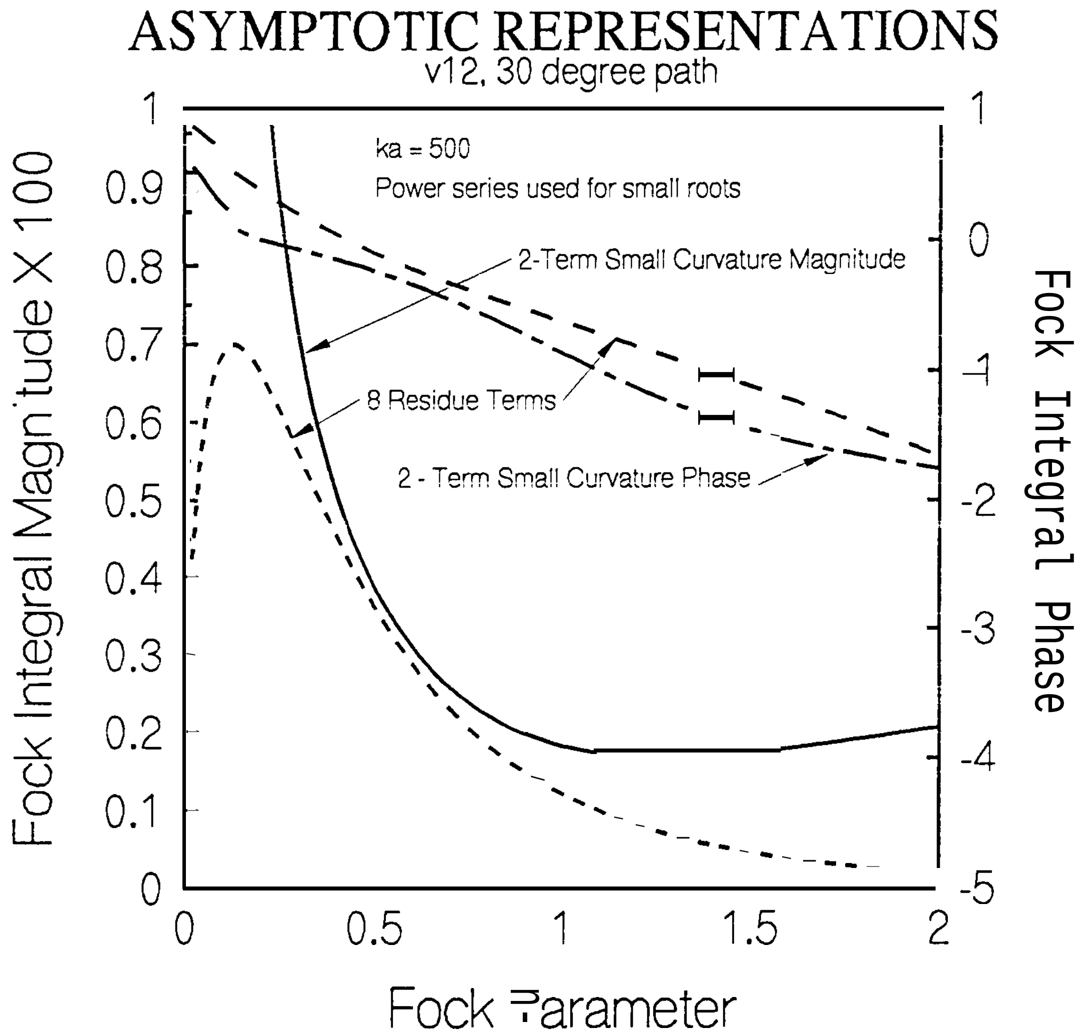


ASYMPTOTIC REPRESENTATIONS
v22, 45 degree path
ka = 500
Power series used for small roots
2-Term Small Curvature Magnitude
8 Residue Terms
2 - Term Small Curvature Phase
Fock Integral Magnitude X 100
Fock Integral Phase
Fock Parameter
0.5
0.4
0.3
0.2
0.1
0
0.5
1
1.5
2
3
2
1
o
-1
-2
-3

ASYMPTOTIC REPRESENTATIONS

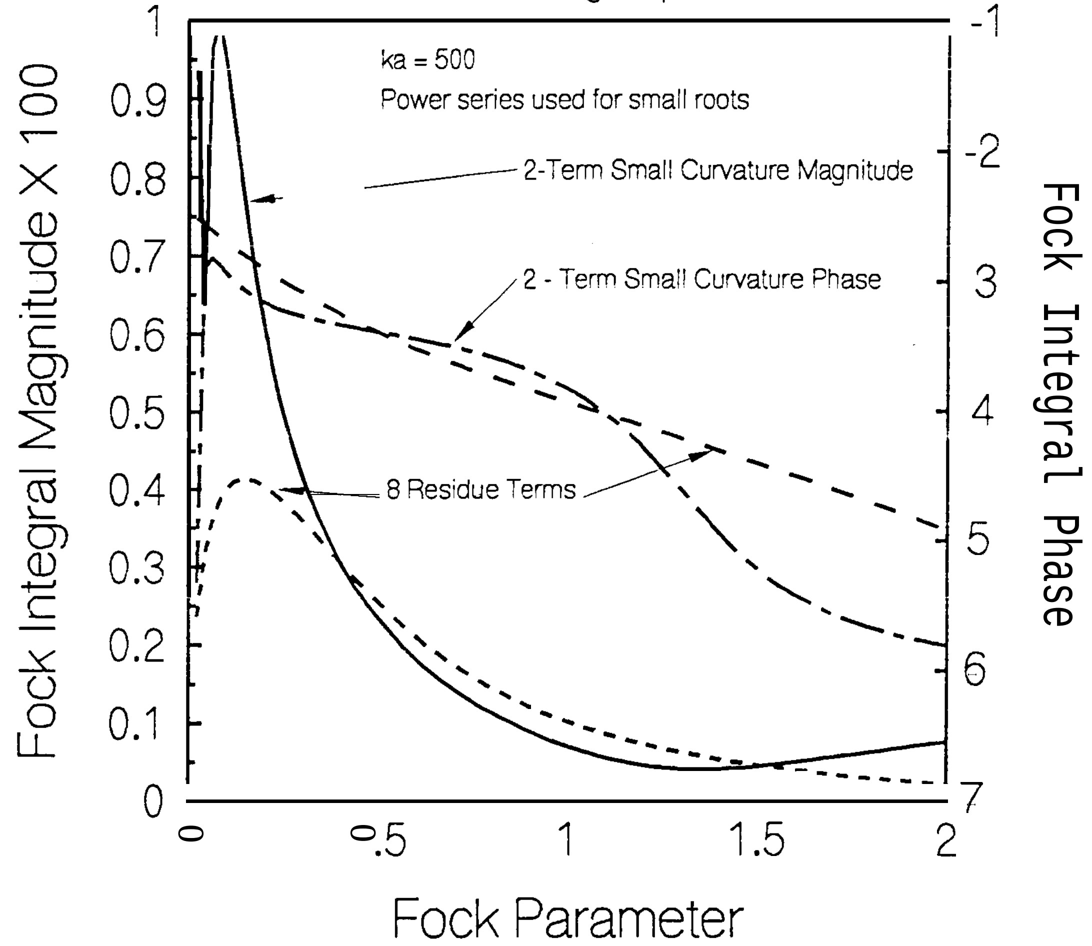
v11, 30 degree path

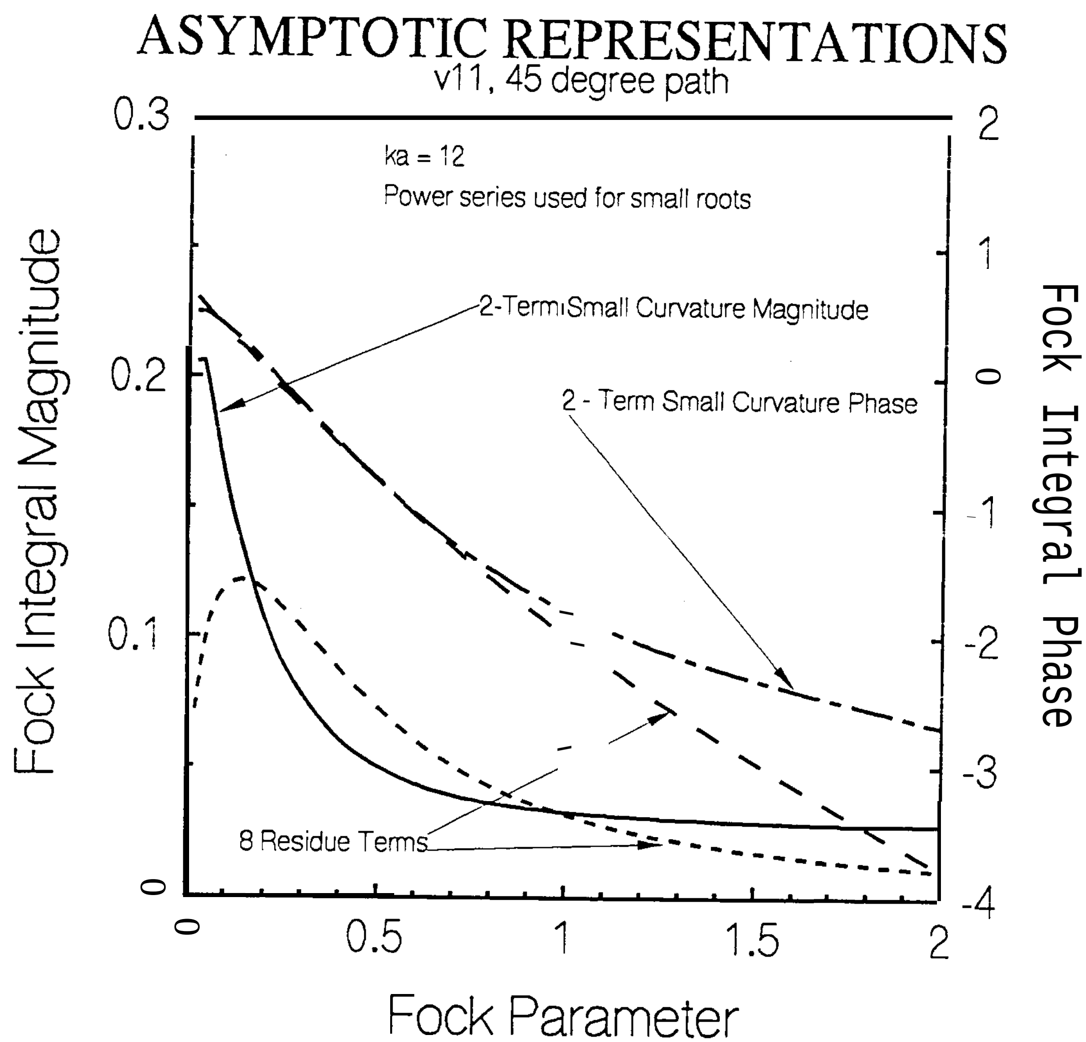



ASYMPTOTIC REPRESENTATIONS
v12, 30 degree path
ka = 500
Power series used for small roots
2-Term Small Curvature Magnitude
8 Residue Terms
2 - Term Small Curvature Phase
Fock Integral Magn'tude X 100
Fock Integral Phase
Fock Parameter

ASYMPTOTIC REPRESENTATIONS

v22, 30 degree path

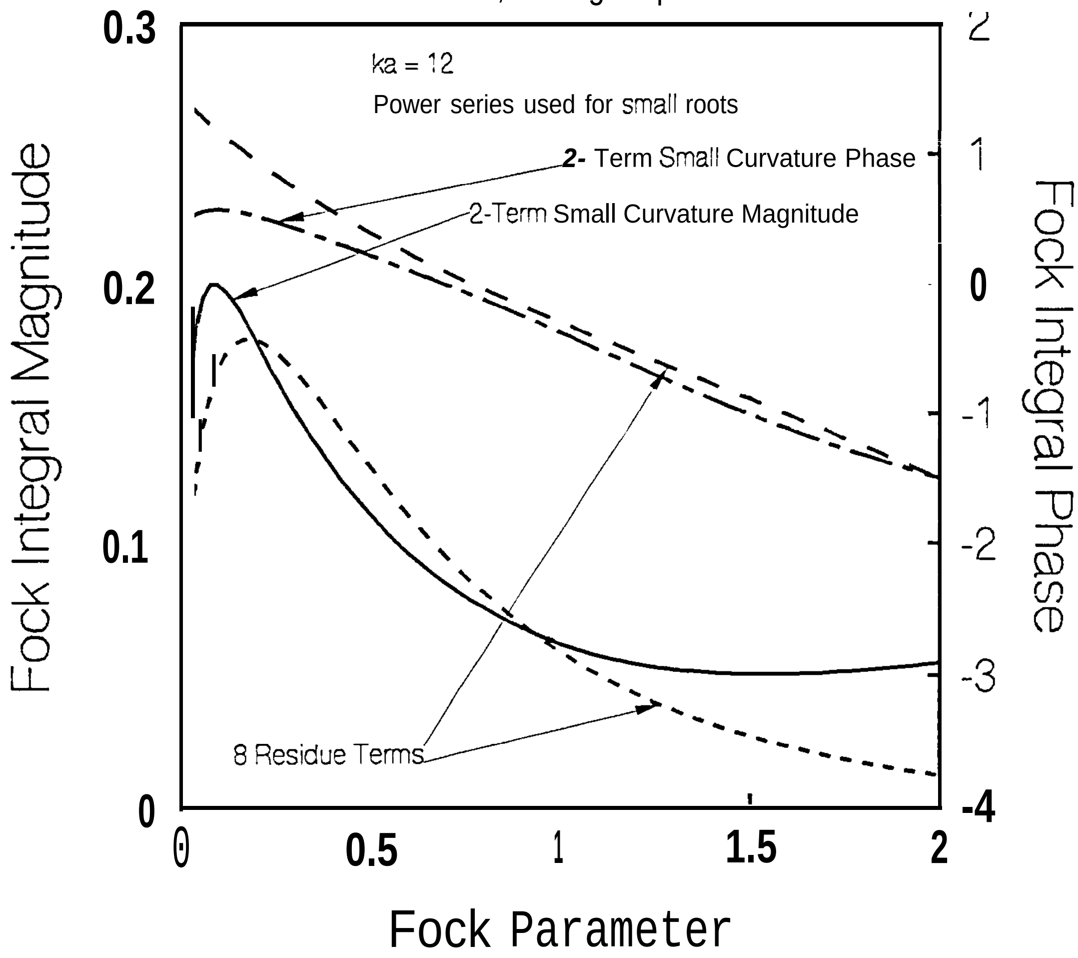




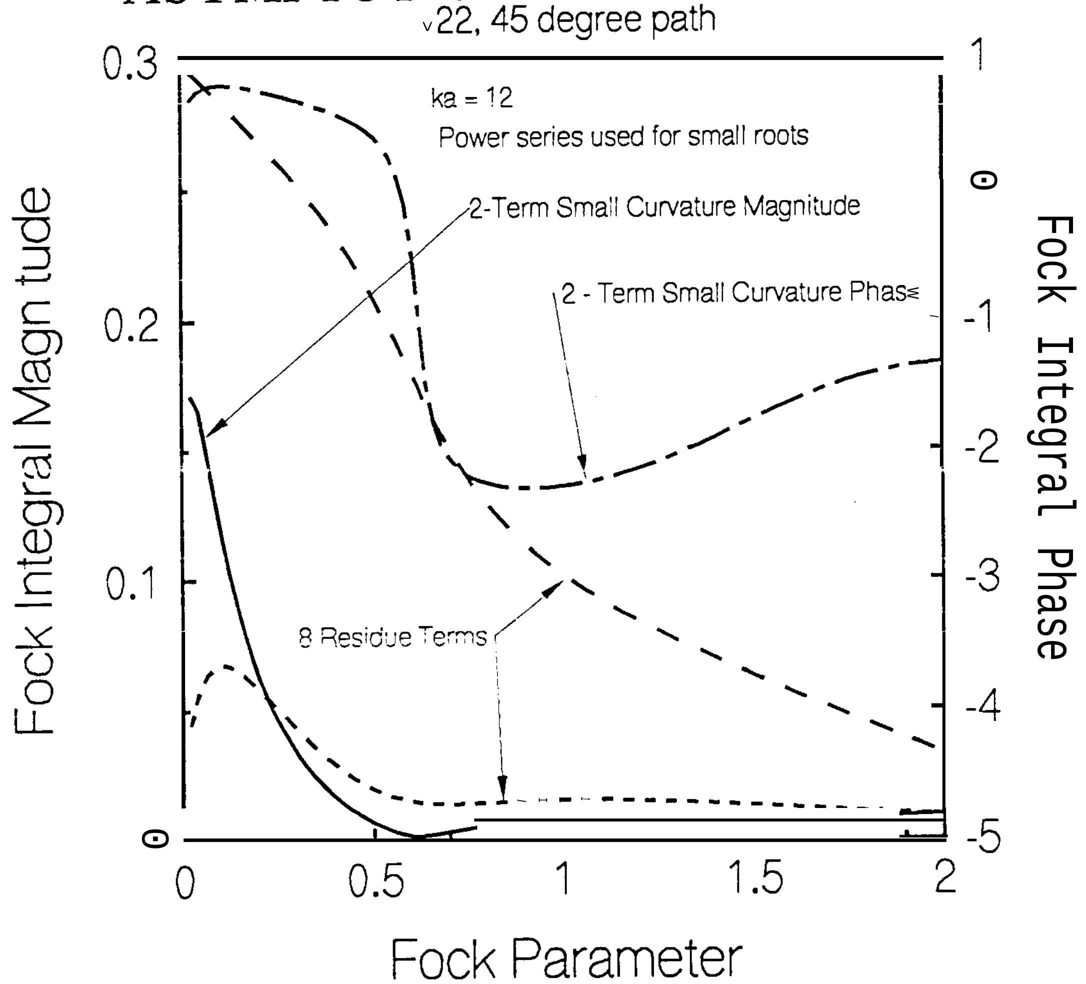
ASYMPTOTIC REPRESENTATIONS
v11, 45 degree path
ka = 12
Power series used for small roots
2-Term Small Curvature Magnitude
2 - Term Small Curvature Phase
8 Residue Terms
Fock Integral Magnitude
Fock Integral Phase
Fock Parameter
0.3
0.2
0.1
0
2
1
0
-1
-2
-3
-4
0.5
1.5

ASYMPTOTIC REPRESENTATIONS

VI 2, 45 degree path

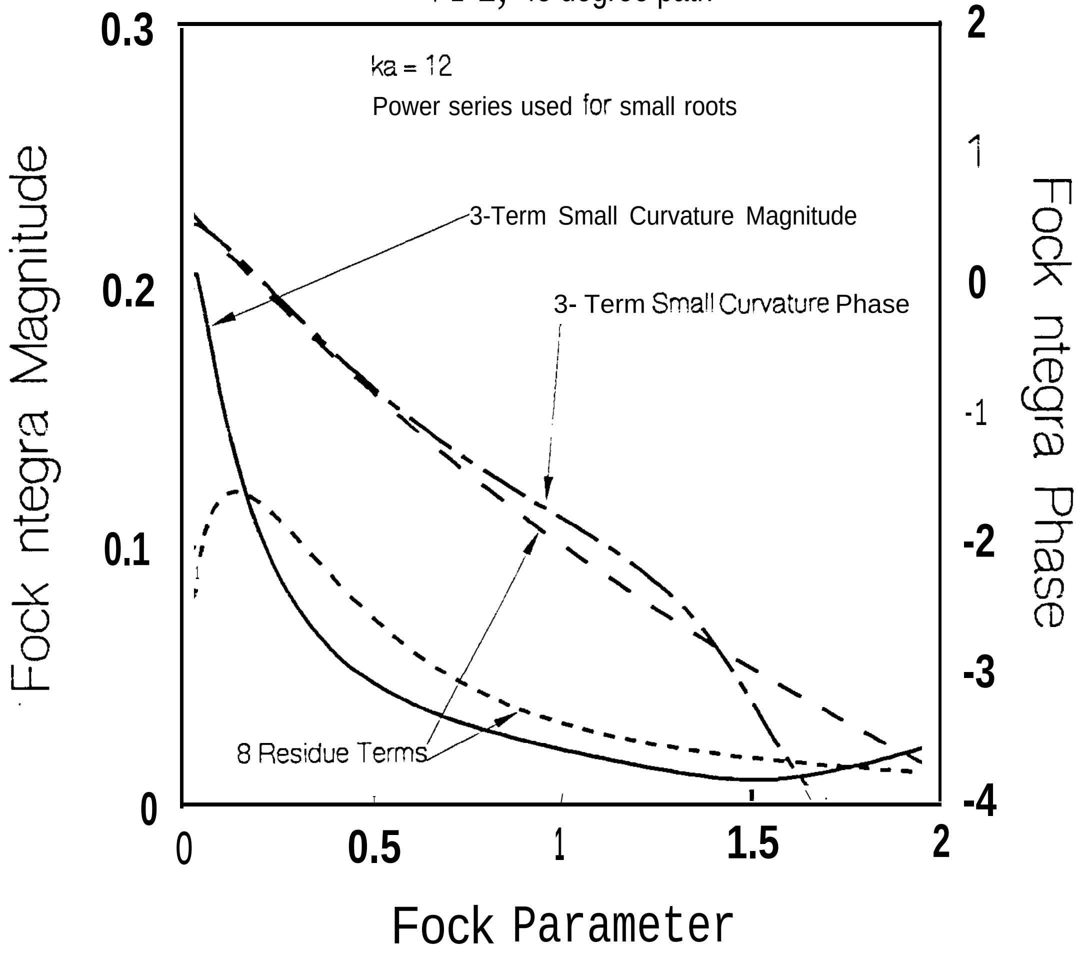


ASYMPTOTIC REPRESENTATIONS



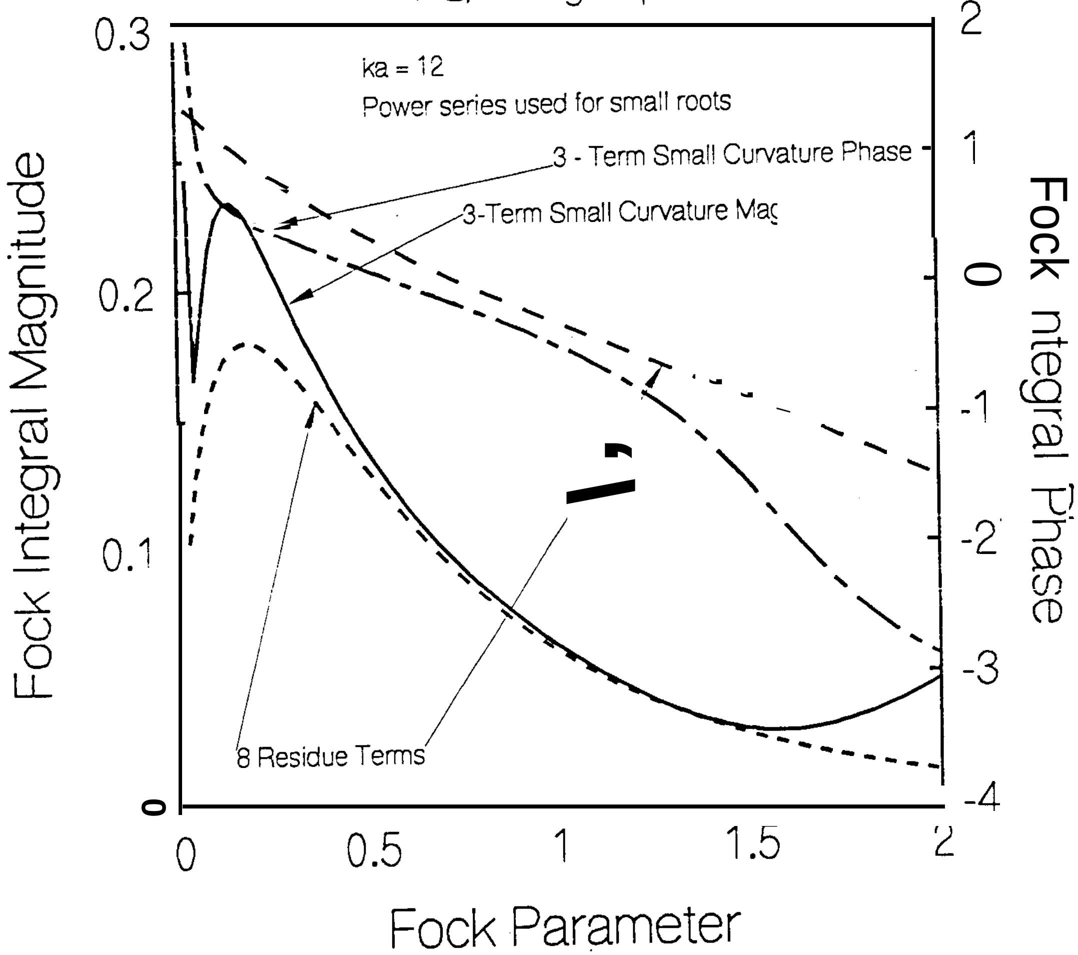
ASYMPTOTIC REPRESENTATIONS

VI 1, 45 degree path



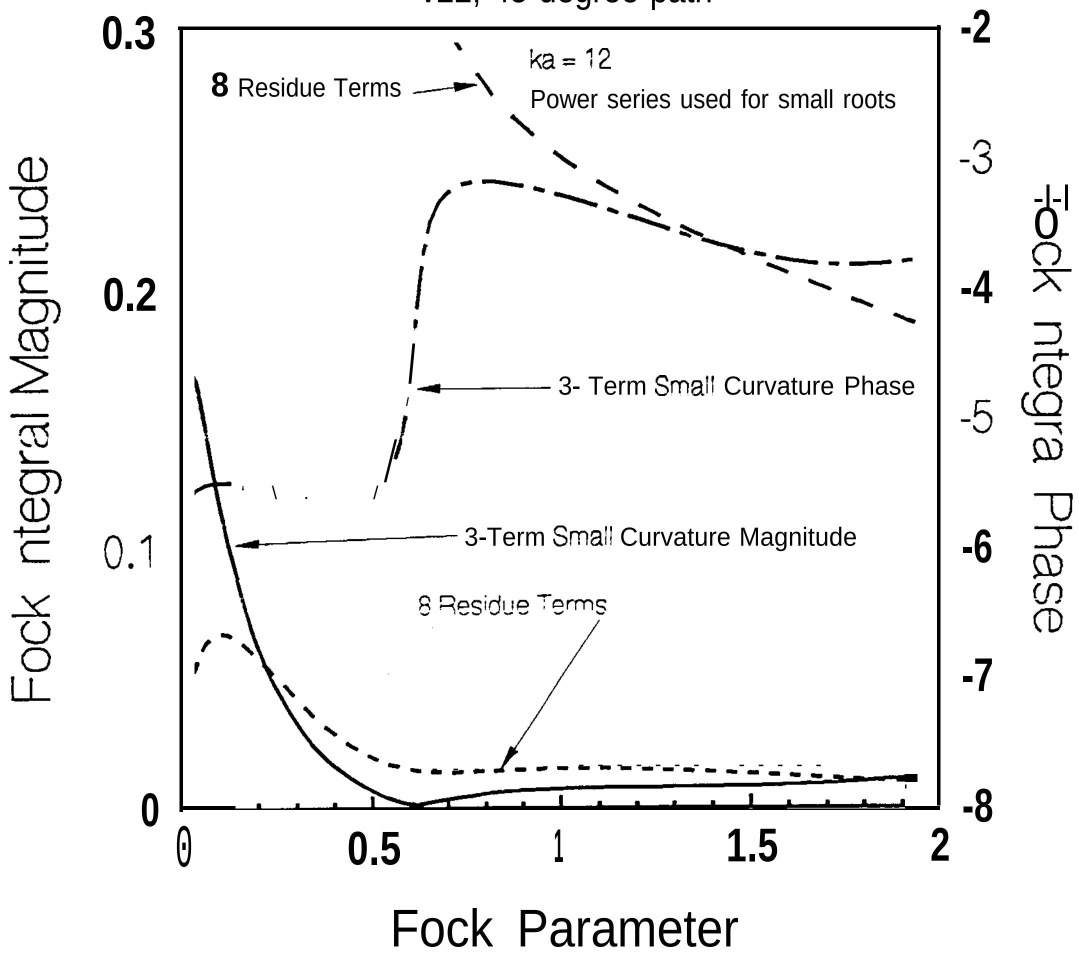
ASYMPTOTIC REPRESENTATIONS

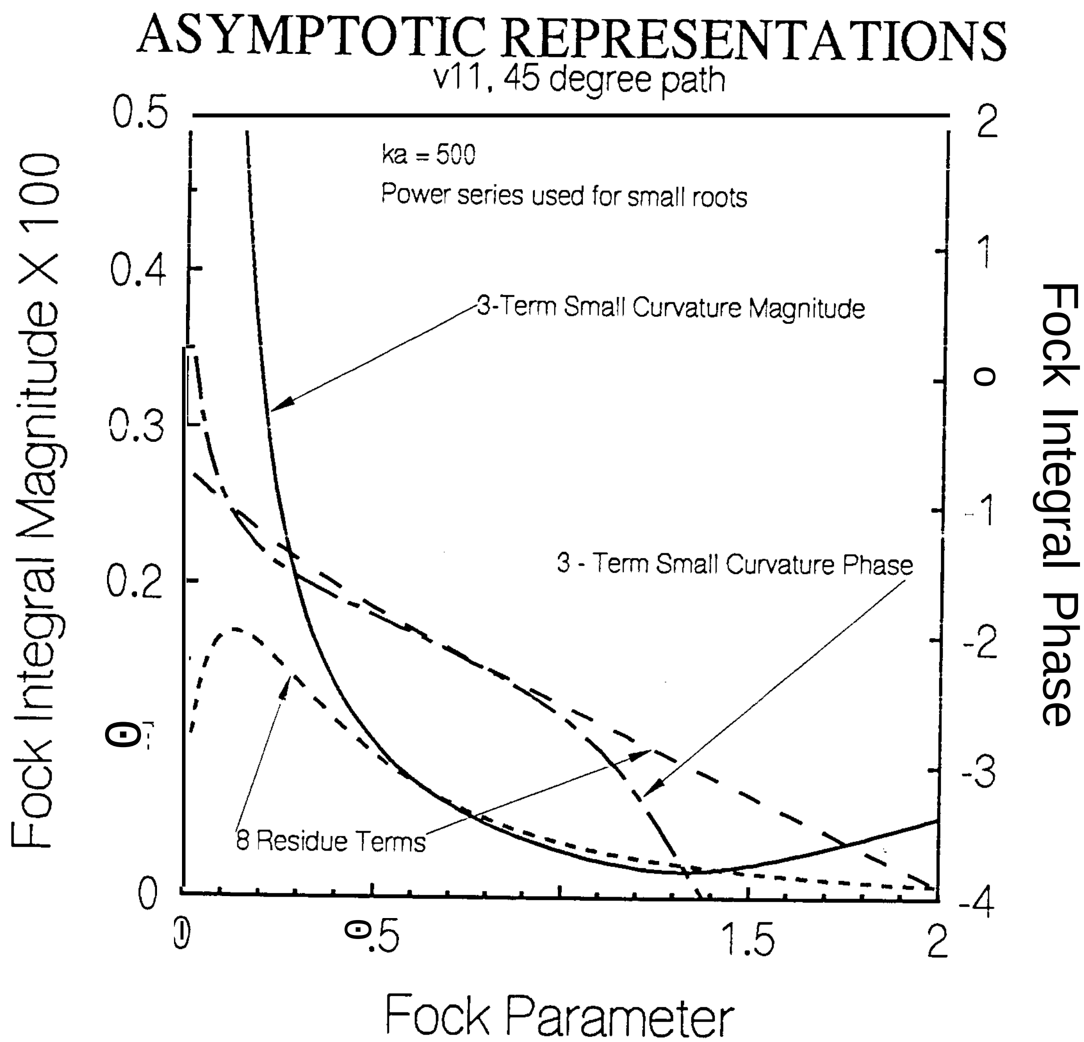
v12, 45 degree path



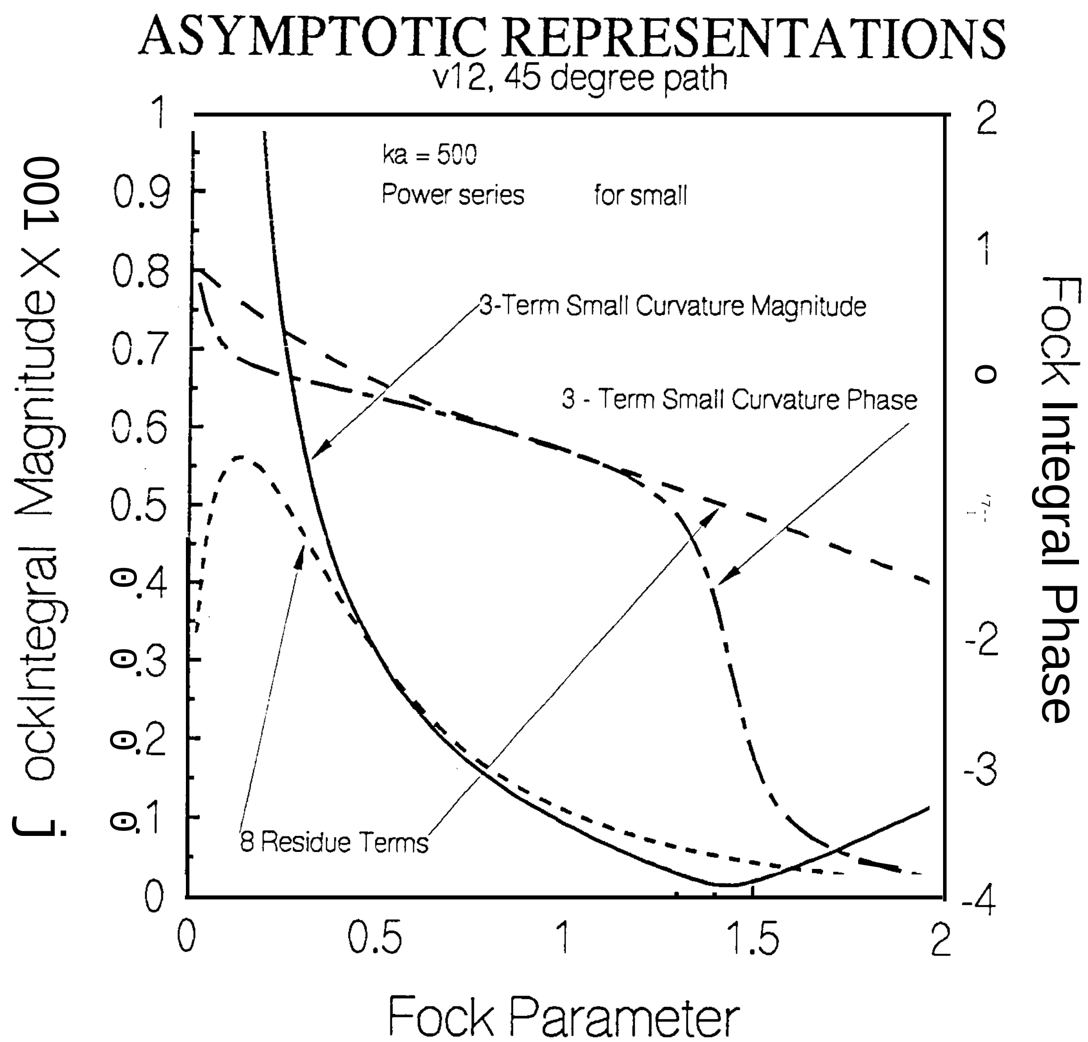
ASYMPTOTIC REPRESENTATIONS

v22, 45 degree path



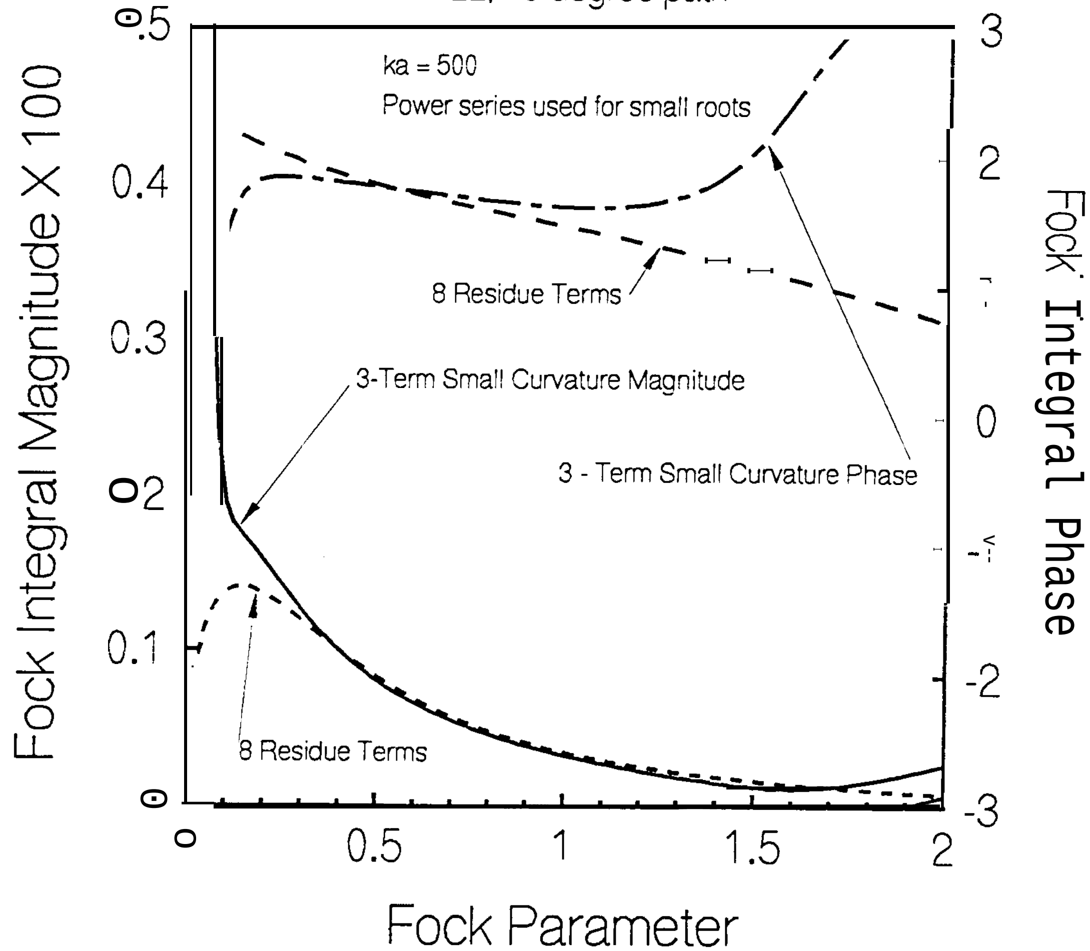


ASYMPTOTIC REPRESENTATIONS
v11, 45 degree path
ka = 500
Power series used for small roots
3-Term Small Curvature Magnitude
3 - Term Small Curvature Phase
8 Residue Terms
Fock Integral Magnitude X 100
Fock Integral Phase
Fock Parameter


ASYMPTOTIC REPRESENTATIONS
v12, 45 degree path
ka = 500
Power series for small
3-Term Small Curvature Magnitude
3 - Term Small Curvature Phase
8 Residue Terms
Fock Parameter
Fock Integral Phase
ockIntegral Magnitude X 100

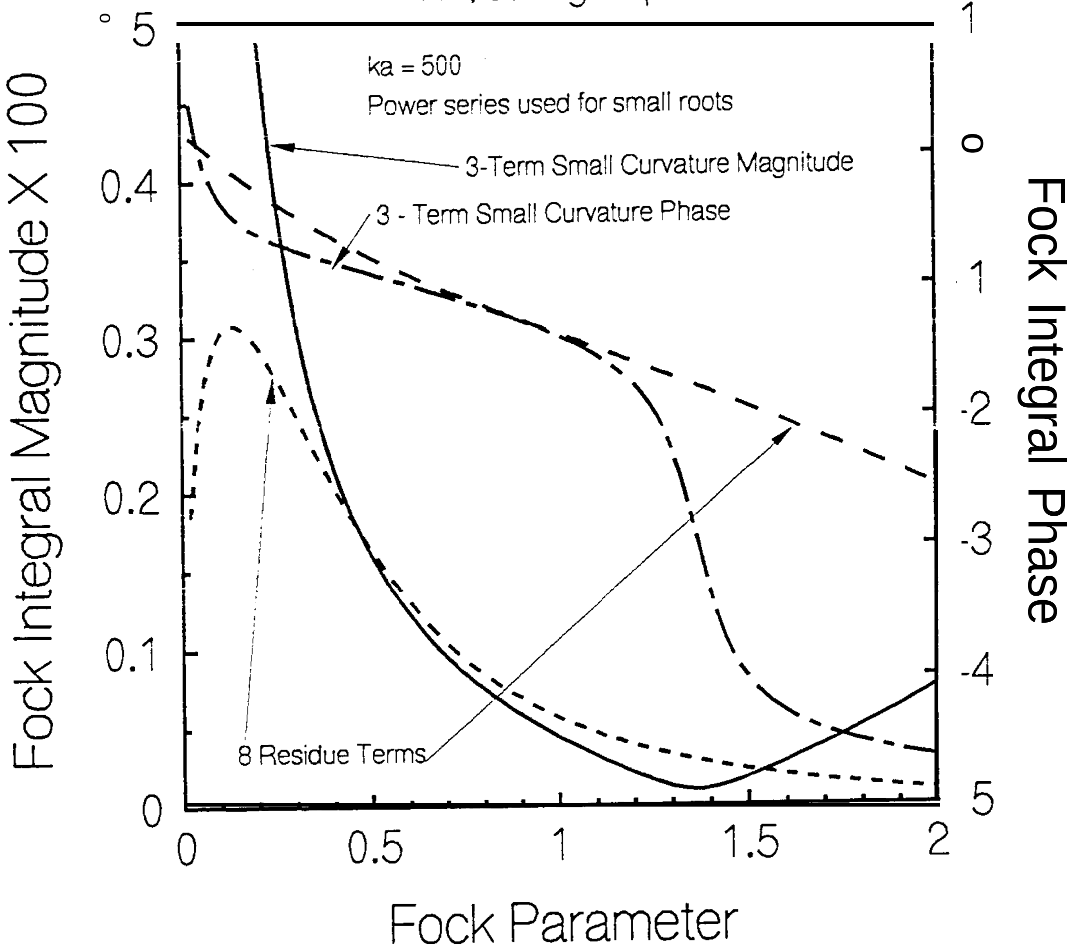
ASYMPTOTIC REPRESENTATIONS

V22, 45 degree path



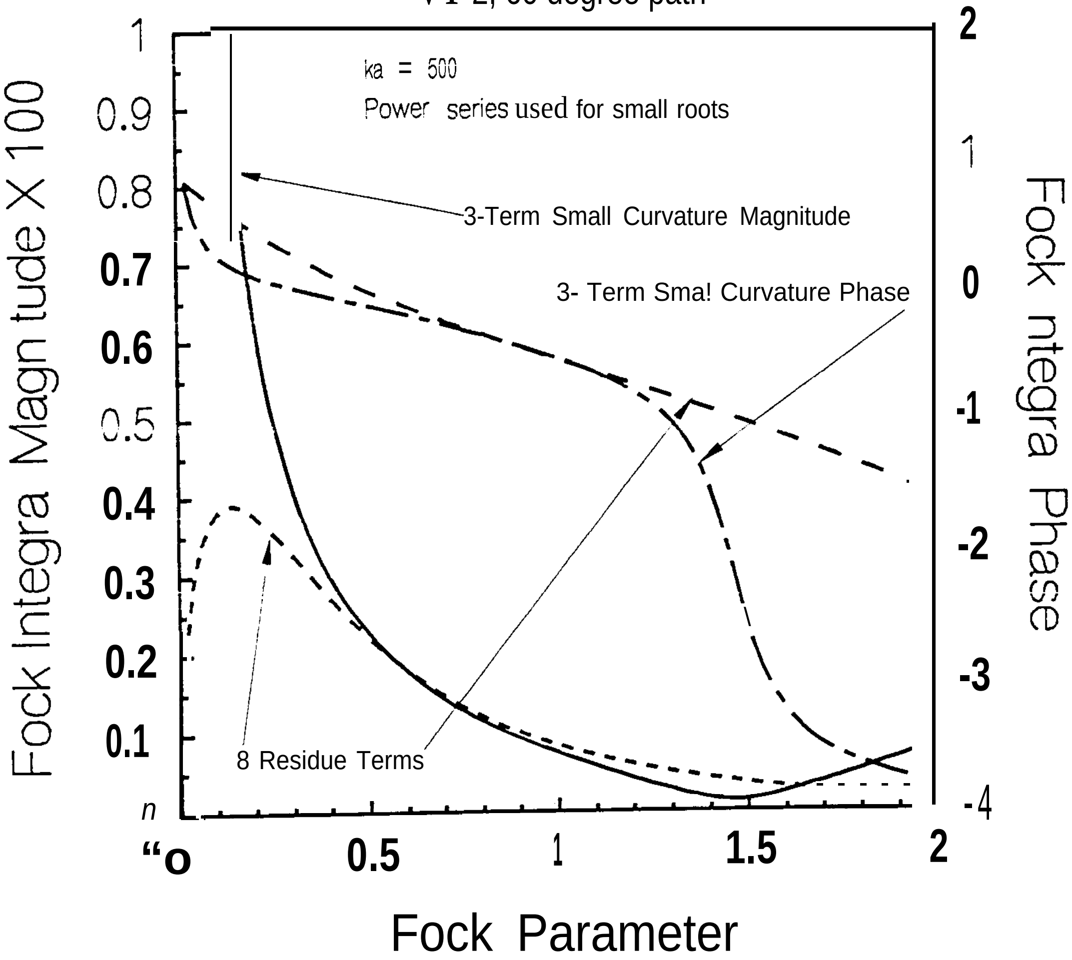
ASYMPTOTIC REPRESENTATIONS

v11, 60 degree path



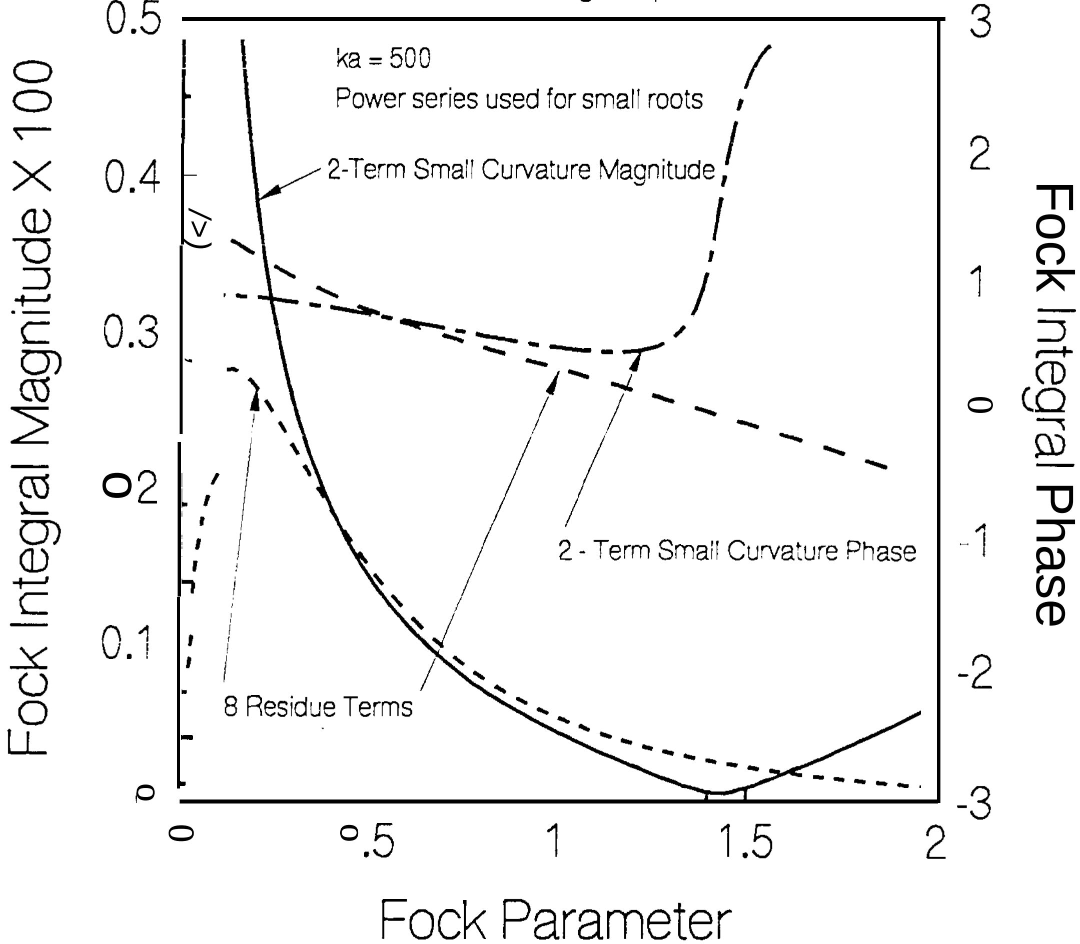
ASYMPTOTIC REPRESENTATIONS

VI 2, 60 degree path



ASYMPTOTIC REPRESENTATIONS

v22, 60 degree path



Concluding Summary

- In the azimuthal case:
  - The small ξ power series is useful only for small q.
  - The small curvature approximation provides good results for large q (and small and moderate ξ).
  - The residue series covers the large ξ regime.
- All of the above can be generalized to the non-azimuthal case.
- The non-azimuthal case has been treated here by reducing the integrals to a sums of integrals of the azimuthal case form.